Approach to ground state and time-independent photon bound for massless spin-boson models

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Abstract: It is widely believed that an atom interacting with the electromagnetic field (with total initial energy well-below the ionization threshold) relaxes to its ground state while its excess energy is emitted as radiation. Hence, for large times, the state of the atom+field system should consist of the atom in its ground state, and a few free photons that travel off to spatial infinity. Mathematically, this picture is captured by the notion of *asymptotic completeness*. Despite some recent progress on the spectral theory of such systems, a proof of relaxation to the ground state and asymptotic completeness was/is still missing, except in some special cases (massive photons, small perturbations of harmonic potentials). In this paper, we partially fill this gap by proving relaxation to an invariant state in the case where the atom is modelled by a finite-level system. If the coupling to the field is sufficiently infrared-regular so that the coupled system admits a ground state, then this invariant state necessarily corresponds to the ground state. Assuming slightly more infrared regularity, we show that the number of emitted photons remains bounded in time. We hope that these results bring a proof of asymptotic completeness within reach.

1 Model and result

1.1 Introduction

This paper fits into a broader project of rigorously controlling interacting Hamiltonian systems that exhibit irreversible behavior. From a more concrete point of view, the problem treated here is inspired by nonrelativistic QED which, in the last two decades, has proven to be a fruitful testing ground for mathematical techniques in quantum field theory (for an overview, see the book [29]) Sacrificing precision for the time being, the setup is as follows. We consider a model of an atom interacting with the electromagnetic field (or, a scalar field, as we will assume in this paper for the sake of simplicity). The field is described by the Hamiltonian $H_{\rm F}$ on Hilbert space $\mathscr{H}_{\rm F}$, it is non-interacting and describes freely propagating scalar bosons with a linear dispersion law (the polarization of the photons does not play a crucial role in the physical problem, hence we omit it from our model). The joint dynamical system (atom + field) is described by a Hamiltonian H on a Hilbert space $\mathscr{H}$ and the field is coupled in a nontrivial way to the atom. Furthermore, we assume that the atom cannot be ionized, i.e. in the absence of coupling its spectrum is discrete. We wish to address the long time behavior of the system.

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1.1.1 Approach to a stationary state

Let O be a local observable in $\mathscr{B}(\mathscr{H})$ (i.e. bounded operators on $\mathscr{H}$) that should be thought of as probing the atom and the field in its spatial vicinity. Let $\Psi_0 \in \mathscr{H}, \|\Psi_0\| = 1$ be the initial state vector (0 refers to time $t = 0$) and $\Psi_t := \mathrm{e}^{-\mathrm{i}tH}\Psi_0$ is the time-evolved state vector. We consider the expectation value ($\langle\cdot,\cdot\rangle$ is the scalar product)

$$\langle O\rangle_t := \langle\Psi_t, O\Psi_t\rangle \tag{1.1}$$

As explained in the abstract, we substantiate the claim that this expression converges to an asymptotic value, as $t \to \infty$. i.e. $\langle O\rangle_t \to \langle O\rangle_\infty$ and that $\langle O\rangle_\infty$ is independent of Ψ_0. If the Hamiltonian H admits a ground state Ψ_{gs}, i.e. $E_{\mathrm{gs}} = \inf\sigma(H)$ is an eigenvalue, then this implies that $\langle O\rangle_\infty = \langle\Psi_{\mathrm{gs}}, O\Psi_{\mathrm{gs}}\rangle$ since one can choose $\Psi_0 = \Psi_{\mathrm{gs}}$. In particular, this means that the eigenvalue E_{gs} is simple and that H cannot have any other eigenvalues. It is this last claim that has been established up to now in great generality. In fact, one can even prove that H has absolutely continuous spectrum, apart from the simple eigenvalue E_{gs}, see [3, 12, 13, 5]. However, this seems not sufficient to prove that (1.1) converges as $t \to \infty$, because $\mathrm{e}^{-\mathrm{i}tH}$ appears twice in this expression (!). Let us immediately add that we find it not at all inconceivable that there is some easy way around this problem, allowing to apply the techniques used for the spectral analysis of H to determine the asymptotics of (1.1). However, this is not the strategy of the present paper. We prove the convergence to an asymptotic value, $\langle O\rangle_t \to \langle O\rangle_\infty$ not by spectral considerations, but by exhibiting explicitly the irreversible density matrix evolution $|\Psi_0\rangle\langle\Psi_0| \to |\Psi_t\rangle\langle\Psi_t|$. If H is too infrared-singular to admit a ground state, then the convergence $\langle O\rangle_t \to \langle O\rangle_\infty$ can still hold provided that the observable O 'does not see' the low-energetic photons. In that case the asymptotic value $\langle O\rangle_\infty$ is a state (a positive, normalized functional in O) that is not of the form $\langle O\rangle_\infty = \langle\Psi_{\mathrm{gs}}, O\Psi_{\mathrm{gs}}\rangle$. From the point of view of our technique, this case is no different from the infrared-regular case.

Finally, we mention that the problem of 'return to equilibrium' at positive temperature has been studied with much more success since in that case, the problem can indeed be reduced to the study of the spectrum of an operator - the so-called *standard Liouvillian* - acting on an appropriate Hilbert space, see [7] for references. For our technique, there is no difference between zero and positive temperature, and the present result on approach to a stationary state was, up to some irrelevant details, in fact already contained in [7].

1.1.2 Scattering theory

Let us again assume that H admits a ground state Ψ_{gs} (and no other eigenstates). The intuition is that the evolved state vector Ψ_t should, at large times t, look like the ground state with a few free photons that travel off to infinity.

To make this intuition more precise, one introduces an identification operator $J_{\mathrm{id}} : \mathscr{H}_{\mathrm{F}} \to \mathscr{H}$, such that $J_{\mathrm{id}}\Omega = \Psi_{\mathrm{gs}}$ with Ω the field vacuum and in general J_{id} maps photon state vectors Ψ_F into state vectors that we think of as 'ground state together with a free photon wavepacket Ψ_F'. One defines the asymptotic wave operators W^+, W^- by

$$W^\pm\Psi_F := \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}t(H-E_{\mathrm{gs}})} J_{\mathrm{id}} \mathrm{e}^{-\mathrm{i}tH_{\mathrm{F}}}\Psi_F \tag{1.2}$$

for Ψ_F in a dense domain. Asymptotic completeness (AC) asserts that the operators $W^\pm : \mathscr{H}_{\mathrm{F}} \to \mathscr{H}$ exist and are unitary. The main unproven aspect of this statement is $\mathrm{Ran}W^\pm = \mathscr{H}$. Indeed, what can happen in the presence of (massless) photons is that $\Psi_t = \mathrm{e}^{-\mathrm{i}tH}\Psi_0$ will contain ever more (as $t \to \infty$) photons with ever smaller energies. But if this is the case then $\Psi_0 = \mathrm{e}^{\mathrm{i}tH}\Psi_t$ can clearly not equal $W^+\Psi_F$ for any $\Psi_F \in \mathscr{H}_{\mathrm{F}}$. For this reason it is important to bound the number of photons of the state vector Ψ_t. Indeed, if one eliminates the soft photons, either by making them massive or by introducing a sharp infrared cutoff in the coupling, then all questions can be answered and in particular AC follows, see [11, 16]. Our result provides a strong exponential bound of the form $\langle\Psi_t, \mathrm{e}^{\kappa N}\Psi_t\rangle \leq C$ with κ a small positive constant and C independent of time. We believe that with this bound as an input, a proof of asymptotic completeness is within reach (see e.g. [15]) and we hope to pursue this in a subsequent paper. Apart from the case of massive bosons, asymptotic completeness can be established if the particle is a harmonic oscillator and the coupling to the field is linear, so that the full Hamiltonian can be explicitly diagonalized [2]. Small perturbations (due to small anharmonicities in the particle potential) can be handled by an expansion introduced in [23], see [28]. Of course, if one considers models not described by quantum field theory, for example; N-particle quantum mechanics, more results are available and we refer to [11] for more references and background.

1.2 Setup

Let $\mathscr{H}_{\rm S}$ be a Hilbert space (modeling the small system) with a self-adjoint Hamiltonian $H_{\rm S}$. The field is given by the one-particle dispersion relation $|q|$ and the Hamiltonian of the whole field is given by

$$H_{\rm F} := \int_{\mathbb{R}^d} \mathrm{d}q |q| a_q^* a_q \tag{1.3}$$

acting on the bosonic (symmetric) Fock space $\mathscr{H}_{\rm F} = \Gamma(\mathfrak{h})$ with $\mathfrak{h} = L^2(\mathbb{R}^d)$ the one-particle space. Here a_q^*, a_q are the creation/annihilation operators (actually, operator-valued distributions) of a mode with momentum $q \in \mathbb{R}^d$ satisfying the canonical commutation relations $[a_q, a_{q'}^*] = \delta(q - q')$. We refer to e.g. [9] for a review of these notions and precise definitions. The Hilbert space of the total system consisting of small system and field, is $\mathscr{H} = \mathscr{H}_{\rm S} \otimes \mathscr{H}_{\rm F}$, and we simply write $H_{\rm S}$ and $H_{\rm F}$ for the operators $H_{\rm S} \otimes 1$ and $1 \otimes H_{\rm F}$ acting on $\mathscr{H}$. For simplicity, the coupling between field and the small system is assumed to be linear in the creation and annihilation operators and of the form $\lambda H_{\rm I}$, with $\lambda \in \mathbb{R}$ a (small) coupling constant and

$$H_{\rm I} = D \otimes \int_{\mathbb{R}^d} \mathrm{d}q \left(\phi(q) a_q^* + \overline{\phi(q)} a_q \right), \tag{1.4}$$

for some $\phi \in \mathfrak{h}$ and Hermitian matrix $D = D^* \in \mathscr{B}(\mathscr{H}_{\rm S})$. Our technique works equally well if one considers a finite sum of such interaction terms, or if one adds a sufficiently small quadratic interaction term, but we prefer to keep the setup as elementary as possible. The formal total Hamiltonian of the system is hence

$$H := H_{\rm S} + H_{\rm F} + \lambda H_{\rm I}, \qquad \text{on } \mathscr{H}. \tag{1.5}$$

To construct H rigorously, we assume throughout that

$$\langle \phi, (1 + \frac{1}{|q|})\phi \rangle_{\mathfrak{h}} = \int_{\mathbb{R}^d} \mathrm{d}q (1 + \frac{1}{|q|}) |\phi(q)|^2 < \infty. \tag{1.6}$$

which yields, by a standard estimate:

$$\|H_{\rm I}\Psi\|^2 \leq 2\|D\| \langle \phi, (1 + 1/|q|)\phi \rangle_{\mathfrak{h}} \, \langle \Psi, H_{\rm F}\Psi \rangle \leq C \langle \Psi, H_{\rm F}\Psi \rangle, \qquad \text{for } \Psi \in \mathrm{Dom}(H_{\rm F}) \tag{1.7}$$

It follows that $H_{\rm I}$ is an infinitesimal perturbation of $H_{\rm F}$ and by the Kato-Rellich theorem, the Hamiltonian H is self-adjoint on $\mathrm{Dom}(H_{\rm S} + H_{\rm F})$.

The following assumption is the key ingredient of our analysis, as it expresses that correlations of the free field vanish in time sufficiently fast.

Assumption A (Decay of correlation functions)**.**

$$\int_{\mathbb{R}_+} \mathrm{d}t \, (1 + t)^\alpha |h(t)| < \infty, \qquad \textit{with } h(t) := \int_{\mathbb{R}^d} \mathrm{d}q \, \mathrm{e}^{-\mathrm{i}t|q|} |\phi(q)|^2, \tag{1.8}$$

for some $\alpha > 0$.

This assumption implies some infrared regularity of the model. In particular, if it is satisfied with $\alpha \geq 1$, then the Hamiltonian H has a ground state[3], as one can establish by, for example, the techniques in [17, 18, 4, 14, 1]. In Section C, we give a condition on the form factor that is sufficient for Assumption A to hold.

The next assumption is meant to exclude situations in which the atom is poorly coupled to the field. In particular, if the function $\phi(q)$ vanishes identically, then one cannot expect relaxation to the ground state, and this should surely be excluded. We assume that $\sigma(H_{\rm S})$, the spectrum of $H_{\rm S}$, is nondegenerate, and let P_e for $e \in \sigma(H_{\rm S})$ be the corresponding (one-dimensional) spectral projectors. We introduce the nonnegative numbers

$$j(e, e') := \mathrm{Tr}[P_e D P_{e'} D P_e] \, \hat{h}(e - e'), \tag{1.9}$$

[3]In fact, weaker conditions suffice for the existence of a ground state. For example, discontinuities in the form factor ϕ can invalidate our results (see Section 1.5.1), but the existence of the ground state depends solely on the behavior of ϕ near 0.

where $\hat{h}(\varepsilon) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}t\varepsilon} h(t)$ is well-defined by virtue of Assumption A. It can be written in a more intuitive fashion as $2\pi \int_{\mathbb{R}^d} \mathrm{d}q\, \delta(|q| - (e - e'))\, |\phi(q)|^2$, which one recognizes as the textbook Fermi Golden Rule expression for a scattering rate. One deduces that $j(e, e') = 0$ whenever $e' \geq e$. Physically, this expresses that the field is in the vacuum state and it can only absorb (and not emit) energy.

Assumption B (Fermi Golden Rule)**.** *We assume that the spectrum of H_{S} is non-degenerate (all eigenvalues are simple) and we let $e_0 := \min \sigma(H_{\mathrm{S}})$ (atomic ground state energy). Most importantly, we assume that for any eigenvalue $e \in \sigma(H_{\mathrm{S}}), e \neq e_0$, there is a sequence $e(i), i = 1, \ldots, n$ of eigenvalues such that*

$$e = e(1) > e(2) > \ldots > e(n) = e_0, \qquad \textit{and} \qquad \forall i = 1, \ldots, n-1 : j(e(i), e(i+1)) > 0 \tag{1.10}$$

with $j(\cdot, \cdot)$ as defined above.

The numbers $j(e, e')$ should be viewed as 'jump rates': We define the one-dimensional spectral projector $P_\Omega = |\Omega\rangle\langle\Omega| \in \mathscr{B}_1(\mathscr{H}_{\mathrm{F}})$ with Ω the vacuum vector in the Fock space $\mathscr{H}_{\mathrm{F}}$. If the joint atom-field system is described by the density matrix $\rho_0 = P_e \otimes P_\Omega$ at time $t = 0$, then formal perturbation theory (Fermi Golden Rule) suggests that the probability to find the atom in state $e' \neq e$ at a later time $t > 0$, is

$$\mathrm{Tr}[P_{e'} \mathrm{e}^{-\mathrm{i}tH} \rho_0 \mathrm{e}^{\mathrm{i}tH}] = j(e, e')(\lambda^2 t) + \mathcal{O}((\lambda^2 t)^2) \tag{1.11}$$

The rigorous version of this formula, given in Proposition 3.3, is a crucial ingredient of our analysis.

1.3 Initial states and observables

We now define the class of initial states ρ_0 and observables O that we consider. For $\psi \in \mathfrak{h}$, let $\mathcal{W}(\psi) \in \mathscr{B}(\mathscr{H}_{\mathrm{F}})$ be the Weyl operator

$$\mathcal{W}(\psi) = \mathrm{e}^{\mathrm{i}\Phi(\psi)}, \qquad \Phi(\psi) := \int dq \left(\psi(q) a_q^* + \overline{\psi(q)} a_q \right) \tag{1.12}$$

We use $\ltimes, \rtimes$ as labels to denote objects characterizing the initial state ('left boundary') and observable ('right boundary'). We pick $\psi_\ltimes, \psi_\rtimes \in \mathfrak{h}$, an atom observable $O_{\mathrm{S}} \in \mathscr{B}(\mathscr{H}_{\mathrm{S}})$ and a density matrix $\rho_{\mathrm{S},0} \in \mathscr{B}_1(\mathscr{H}_{\mathrm{S}})$, i.e. such that $\mathrm{Tr}\, \rho_{\mathrm{S},0} = 1$ and $\rho_{\mathrm{S},0} \geq 0$. Then we put

$$O := O_{\mathrm{S}} \otimes \mathcal{W}(\psi_\rtimes), \qquad \rho_0 := \rho_{\mathrm{S},0} \otimes \mathcal{W}(\psi_\ltimes) P_\Omega \mathcal{W}^*(\psi_\ltimes). \tag{1.13}$$

If $\rho_{\mathrm{S},0} = |\psi_{\mathrm{S},0}\rangle\langle\psi_{\mathrm{S},0}|$ for some $\psi_{\mathrm{S},0} \in \mathscr{H}_{\mathrm{S}}$, then $\rho_0 = |\Psi_0\rangle\langle\Psi_0|$ with $\Psi_0 = \psi_{\mathrm{S},0} \otimes \mathcal{W}(\psi_\ltimes)\Omega$ and this is assumed for simplicity in the next section (for notational convenience, we use the general case in later sections). We need to assume some regularity properties on $\psi_\ltimes, \psi_\rtimes$:

Assumption C (Regularity of initial states and observables)**.**

$$\int_{\mathbb{R}_+} \mathrm{d}t\, (1 + |t|)^\alpha |h_\ltimes(t)| < \infty, \qquad \textit{with} \qquad h_\ltimes(t) := \langle \phi, \mathrm{e}^{-\mathrm{i}|q|t} \psi_\ltimes \rangle_{\mathfrak{h}} \tag{1.14}$$

$$\int_{\mathbb{R}_+} \mathrm{d}t\, (1 + |t|)^\alpha |h_\rtimes(t)| < \infty, \qquad \textit{with} \qquad h_\rtimes(t) := \langle \phi, \mathrm{e}^{\mathrm{i}|q|t} \psi_\rtimes \rangle_{\mathfrak{h}} \tag{1.15}$$

$$\sup_{t \geq 0} |h_\bowtie(t)| (1 + t)^\alpha < \infty, \qquad \textit{with} \qquad h_\bowtie(t) := \langle \psi_\ltimes, \mathrm{e}^{\mathrm{i}|q|t} \psi_\rtimes \rangle_{\mathfrak{h}} \tag{1.16}$$

for $\alpha > 0$.

Throughout our paper, we always assume that Assumptions A and C are satisfied with the same parameter $\alpha > 0$. This is done for the sake of simplicity, though it slightly weakens the result (see Remark 1.4 below).

1.4 Results

Write $\Psi_t := \mathrm{e}^{-\mathrm{i}tH}\Psi_0$ for some initial vector $\Psi_0 \in \mathscr{H}$ and H as defined in (1.5). We define the Weyl algebra $\mathfrak{W}_\alpha, \alpha > 0$ to be the C^*-algebra generated by 'atomic' observables $A \otimes \mathbb{1}$ and Weyl-operators $\mathbb{1} \otimes \mathcal{W}(\psi_{\diagup\!\!\!\diagup})$ with $\psi_{\diagup\!\!\!\diagup} \in \mathfrak{h}$ satisfying (1.15).

Theorem 1.1. *Assume that Assumption A and Assumption B are satisfied. Then, there is a $\lambda_0 > 0$ such that, for any coupling strength $0 < |\lambda| \leq \lambda_0$, the following holds true:*

1) *There is a bounded linear functional $O \mapsto \langle O \rangle_\infty$ on $\mathfrak{W}_\alpha$ such that*

$$\lim_{t\to\infty} \langle \Psi_t, O\Psi_t \rangle = \langle O \rangle_\infty \tag{1.17}$$

for any initial vector $\Psi_0 \in \mathscr{H}$ with $\|\Psi_0\| = 1$ and $O \in \mathfrak{W}_\alpha$.

2) *Let $\Psi^0_{\mathrm{gs}} = \psi_{e_0} \otimes \Omega, \|\Psi^0_{\mathrm{gs}}\| = 1$ be the normalized ground state of the uncoupled Hamiltonian $H_{\mathrm{S}} + H_{\mathrm{F}}$ and let O be of the form* (1.13) *with $\psi_{\diagup\!\!\!\diagup}$ satisfying* (1.15)*, then*

$$\langle O \rangle_\infty - \langle \Psi^0_{\mathrm{gs}}, O\Psi^0_{\mathrm{gs}} \rangle = \mathcal{O}(|\lambda|^{\min(2\alpha,1)}), \qquad \lambda \to 0 \tag{1.18}$$

3) *If O and Ψ_0 (that is: the $\psi_{\ltimes}, \psi_{\diagup\!\!\!\diagup}$ that determine them) satisfy the three bounds of Assumption C, then*

$$|\langle O \rangle_\infty - \langle \Psi_t, O\Psi_t \rangle| \leq \mathcal{O}(t^{-\alpha}), \qquad t \to \infty \tag{1.19}$$

Remark 1.2. *One could be tempted to interpret the functional $\langle \cdot \rangle_\infty$ as the expectation in the ground state of the coupled system, but this is not correct since, for $\alpha < 1$, the coupled system does in general not have a ground state in $\mathscr{H}$ (although a ground state does exist in the Hilbert space corresponding to a different representation of the operator algebra). On the other hand, if the system does admit a ground state $\Psi_{\mathrm{gs}} \in \mathscr{H}$, then by choosing $\Psi_0 = \Psi_{\mathrm{gs}}$,* (1.17) *immediately implies that $\langle O \rangle_\infty = \langle \Psi_{\mathrm{gs}}, O\Psi_{\mathrm{gs}} \rangle$.*

Our second result bounds the number of emitted bosons. Let N be number operator on the Fock space $\mathscr{H}_{\mathrm{F}}$.

Theorem 1.3. *Assume that Assumption A for some $\alpha > 0$ and Assumption B are satisfied. Then, there are $\lambda'_0, \kappa'_0 > 0$ such that, for any coupling strength λ with $0 < |\lambda| \leq \lambda'_0$, complex number κ with $|\kappa| \leq \kappa'_0$, and initial vector Ψ_0 with $\psi_{\ltimes}$ satisfying the bound* (1.14) *in Assumption C, we have*

$$\left|\langle \Psi_t, (\mathbb{1} \otimes \mathrm{e}^{\kappa N})\Psi_t \rangle\right| \leq \breve{C} \exp\left(C|t|^{(1-\min(\alpha,1))}\right), \qquad t \geq 0 \tag{1.20}$$

where the constant $\breve{C}$ depends on Ψ_0, but C does not (and none of them depends on λ, κ or t). In particular, if $\alpha \geq 1$, then the LHS is bounded uniformly in time.

Remark 1.4. *As indicated below Assumption C, we prefer to keep one constant α throughout the paper. Let us describe the possible improvement of Theorem 1.3 if we were to drop this constraint. Assume again that Assumption A holds for some $\alpha > 0$ and assume the the bound* (1.14) *in Assumption C is satisfied for some $\alpha_{\ltimes} > 0$, then the photon number bound* (1.20) *still holds with α determined by Assumption A, regardless of the value of $\alpha_{\ltimes}$. This is clear from the inspection of the last part of the proof of Theorem 1.3 in Section 4.2.3.*

1.5 Discussion of the results

1.5.1 Quadratic Hamiltonians

The easiest way to understand our results and the different conditions involved, is to compare them to an integrable model where the same questions can be asked. Consider the formal Hamiltonian

$$H = \int \mathrm{d}q |q| a^*_q a_q + \int \mathrm{d}q \left(\phi(q) a^*_q + \overline{\phi(q)} a_q\right), \qquad \phi \in \mathfrak{h} \tag{1.21}$$

which fits into our framework by taking $\mathscr{H}_{\rm S} = \mathbb{C}$ (in that case $H_{\rm S}$ is an irrelevant number). By completing the square (which, in this context, is a special case of a 'Bogoliubov transformation') we can rewrite it as

$$H = E_{gs} + \int \mathrm{d}q |q| b_q^* b_q, \qquad b_q := a_q + \frac{\phi(q)}{|q|}, \qquad E_{gs} := -\int \mathrm{d}q \frac{|\phi(q)|^2}{|q|} \tag{1.22}$$

If $\phi \in \mathfrak{h}$ and $\phi/\sqrt{|q|} \in \mathfrak{h}$, then the term linear in a/a^* is an infinitesimal perturbation of the quadratic term (the first term in (1.21)). The operator H is self adjoint on the domain of the quadratic term and it is bounded below by $E_{gs} > -\infty$. If moreover $\phi/|q| \in \mathfrak{h}$, then H has a normalizable ground state, given by

$$\Psi_{gs} := \mathrm{e}^{-\|\phi/|q|\|^2} \mathcal{W}(\phi/|q|)\Omega, \tag{1.23}$$

with the Weyl operator $\mathcal{W}(\psi), \psi \in \mathfrak{h}$ as defined in Section 1.3. We refer to e.g. [8] for an extended and rigorous discussion of quadratic Hamiltonians.

Let us look into ergodic properties of the evolution generated by a quadratic Hamiltonian. Let $\psi_{\diagup} \in L^2$, and consider the observable $O = \mathcal{W}(\psi_{\diagup})$. Then by explicit calculation

$$\langle \Omega, \mathrm{e}^{\mathrm{i}tH} O \mathrm{e}^{-\mathrm{i}tH} \Omega \rangle = \mathrm{e}^{-\|\psi_{\diagup}\|^2} \mathrm{e}^{2\mathrm{i}\mathrm{Re}\,\langle \psi_{\diagup}, (\mathrm{e}^{\mathrm{i}t|q|}-1)\frac{\phi}{|q|}\rangle} \tag{1.24}$$

Recall the correlation function $h_{\diagup}(t) = \langle \psi_{\diagup}, \mathrm{e}^{\mathrm{i}t|q|}\phi \rangle$ from Assumption C. Clearly, if $h_{\diagup} \in L^1(\mathbb{R}, \mathrm{d}t)$, then the RHS of (1.24) converges as $t \to \infty$. In particular, this can be true even when $\phi/|q| \notin \mathfrak{h}$, that is, if H has no ground state. One can easily convince oneself that the $t \to \infty$-asymptotics of (1.24) does not change if we consider a general initial state $\rho_0 = |\Psi_0\rangle\langle\Psi_0|$ with $\Psi_0 = \mathcal{W}(\psi_{\diagup})\Omega, \|\Psi_0\| = 1$ and $\psi_{\diagdown}$ such that $h_{\diagdown}(t) = \langle \phi, \mathrm{e}^{-\mathrm{i}t|q|}\psi_{\diagdown} \rangle$ is integrable and $h_{\bowtie}(t) = \langle \psi_{\diagdown}, \mathrm{e}^{\mathrm{i}t|q|}\psi_{\diagup} \rangle$ vanishes at infinity. Next, we study the number of emitted photons

$$\begin{aligned} \langle \Omega, \mathrm{e}^{\mathrm{i}tH} \mathrm{e}^{\kappa N} \mathrm{e}^{-\mathrm{i}tH} \Omega \rangle &= \langle \Omega, \exp\left(\kappa \int \mathrm{d}k (a_k + \varphi_t(k))^* (a_k + \varphi_t(k)) \right) \Omega \rangle, \qquad \varphi_t := (1 - \mathrm{e}^{\mathrm{i}t|q|}) \frac{\phi}{|q|} && (1.25) \\ &= \exp\left(\|\varphi_t\|^2 (\mathrm{e}^{\kappa} - 1) \right) = \exp\left((\mathrm{e}^{\kappa} - 1) \int_0^t \mathrm{d}s \int_0^t \mathrm{d}s' h(s - s') \right) && (1.26) \end{aligned}$$

It is clear that this expression remains bounded if (and only if) $\phi/|q| \in L^2$, hence if H has a ground state. Moreover, we see that the rate of growth of the LHS of (1.20) in Theorem 1.3 corresponds roughly to estimating $h(s - s')$ by $|h(s - s')|$ in the integral.

1.6 Plan of the proof

Our two results, relaxation to the ground state and the photon bound, are very similar from the technical point of view, even if their physical meaning possibly is not. For this reason, we focus exclusively on the relaxation to the ground state in the present section, and we devote a few words to the photon bound at the end.

The proof relies on the following philosophy. The original problem is formulated as a perturbation (with small parameter λ) of an integrable Hamiltonian $H_{\rm S} + H_{\rm F}$ whose dynamics does not have the phenomenon that we want to exhibit: it does not relax into the ground state. This can already be seen by remarking that the atom S is not coupled to the field F and, as the former is finite-dimensional, its dynamics is oscillatory. However, the Fermi Golden Rule (1.11) provides us with a picture that does capture the dissipative behavior: The fact that the state of the atom changes by jumps between eigenstates of the Hamiltonian $H_{\rm S}$ suggests the following approximation.

$$\rho_t \approx \mathrm{e}^{-\mathrm{i}t\mathrm{ad}(H_{\rm S}) + \lambda^2 t M} \rho_{\rm S,0} \otimes P_\Omega \tag{1.27}$$

where M is the generator of a dissipative dynamics that we can loosely describe as the Markov jump process on eigenstates of $H_{\rm S}$ with jump rates given by (1.11) and exponential decay of the off-diagonal (in $H_{\rm S}$-basis) part of the density matrix. The approximation becomes exact (at least as far as the S-state is concerned), as $\lambda \to 0, t \to \infty$ such that $t\lambda^2$ is held fixed. This was already advocated by Van Hove [19] and it was made precise by Davies [6]. We state it explicitly in Proposition 3.3 and we review the proof in Appendix B.

The underlying physical reason why (1.27) is a good approximation is that P_Ω is invariant under the free F-dynamics and any disturbance in the field caused by S is carried away (disperses) to spatial infinity quickly,

such that it is irrelevant for the further evolution, and one can pretend that the state of the field remains P_Ω. The dispersive property is a consequence of the temporal decay of field correlations for the uncoupled dynamics, which is our Assumption A. It is therefore plausible that the evolution of S is Markovian on time scales longer than the time necessary for a field excitation to disperse away

The approximate dynamics in (1.27) exhibits relaxation to the projection onto the (uncoupled) ground state $\psi_{e_0} \otimes \Omega$ provided that sufficiently many jump rates are nonzero; this is captured by the Fermi Golden Rule Assumption B.

In a nutshell, our strategy is to use the dynamics (1.27) as a zero-order term of our expansion for the full dynamics. Note that our expansion is not simply in powers of the coupling constant λ; the exponent in (1.27) clearly has zeroth and second order contributions in λ. The reason we refer to (1.27) as zeroth order is that all other contributions to the dynamics are small compared to this term, or rather, to its dissipative effect.

Because of the jumps described by M, the dynamics (1.27) is stochastic and hence our task reduces to controlling a small perturbation (the real dynamics at finite but small λ) of a stochastic evolution. This is quite a tractable problem that can be handled by analytic perturbation theory of isolated eigenvalues and a cluster expansion. Similar expansions were developed e.g. in [24, 20] and, very closely to the setup of the present paper, in [26, 7].

The main result of this expansion is that we manage to represent $\mathrm{Tr}\,\rho_t O$ as a one-dimensional polymer gas (this dimension corresponds to time). Then the problem of showing that there is a well-defined and unique asymptotic state is analogous to the problem of proving decay of correlations in the one-dimensional gas.

More precisely, the polymer representation is

$$\mathrm{Tr}(\rho_t O) = k_\ltimes k_\rtimes \sum_{\mathcal{A}} \prod_{A\in\mathcal{A}} v(A), \qquad \text{for } t = n/\lambda^2 \tag{1.28}$$

where the sum is over collections $\mathcal{A}$ of sets $A \subset \{0, 1, \ldots, n, n{+}1\}$ such that for any two sets A_1, A_2, $\mathrm{dist}(A_1, A_2) > 1$ where we write $\mathrm{dist}(A_1, A_2) = \min_{i\in A_1, j\in A_2} |i-j|$. The sets A are called polymers, the numbers $v(A)$ are polymer weights. Moreover, only the $v(A)$ with $0 \in A$, $(n+1) \in A$ depend on the initial state, resp. the observable. To a good approximation (the error made is not important for the present discussion),

$$k_\ltimes \sim \mathrm{Tr}\,\rho_0, \qquad k_\rtimes \sim \mathrm{Tr}[(P_{e_0} \otimes P_\Omega)O] \tag{1.29}$$

Our goal is to prove that

$$\lim_{t\to\infty} \mathrm{Tr}(\rho_t O) = \mathrm{Tr}(\rho_0)\langle O\rangle_\infty \tag{1.30}$$

where $\langle O\rangle_\infty$ does not depend on ρ_0. Setting all $v(A) = 0$ essentially amounts to pretending that $\rho_t = P_{e_0} \otimes P_\Omega$ for all t. In that case $\mathrm{Tr}(\rho_t O) = k_\ltimes k_\rtimes$ and the expectation value of the observable is independent of the initial state, apart from the trivial normalization factor $k_\ltimes$. The polymers A contain corrections to this picture. These corrections originate from the fluctuations of the Markovian dynamics generated by M and from the corrections to the Markovian behavior (1.27). Pictorially, let $A = \{\tau_1, \tau_2, \ldots \tau_m\}$ with $\tau_i \le \tau_{i+1}$, then $v(A)$ describes correlated deviations from the assumption that $\rho_t = P_{e_0} \otimes P_\Omega$ in the time-interval $\mathrm{Dom}(\tau_1), \ldots, \mathrm{Dom}(\tau_{m-1})$, where $\mathrm{Dom}(\tau_m) \equiv (1/\lambda^2)[\tau_m - 1, \tau_m]$. The detailed construction of the polymer representation (1.28) is carried through in Section 2.

To prove decay of correlations, we use a standard cluster expansion, which we review in Appendix A. A possible condition for the applicability of the cluster expansion method is the "Kotecky-Preiss" criterion. Applied to our model, it demands that,

$$\sum_{A:\mathrm{dist}(A,A')\le 1} v(A)\mathrm{e}^{a(A)} \le a(A'), \tag{1.31}$$

with $A, A' \subset \{1, \ldots, n\}$ and $a(\cdot)$ an n-independent positive function on polymers. We formulate (1.31) (together with some other statements) in Lemma 2.4. In our case $a(A) \equiv \epsilon C|A|$ where $\epsilon = |\lambda|^{2\min(\alpha,1)}$ can be seen as a renormalized coupling constant.

To prove that the Kotecky-Preiss criterion (1.31) is satisfied, we use a Dyson (or Duhamel) expansion, it relies on the smallness of the coupling constant λ and the decay of field correlations. This is intuitive; if the correlations of the free field decay roughly as $t^{-(1+\alpha)}$ (cfr. Assumption A) then one could conjecture, for example, that for $A = \{\tau_1, \tau_2\}$ also the weight $v(A)$ decays as $(\mathrm{dist}(\mathrm{Dom}(\tau_1), \mathrm{Dom}(\tau_2)))^{-(1+\alpha)}$ as $\tau_2 - \tau_1 \to \infty$. This picture

turns out to be essentially correct. In fact, we get $|v(\{\tau_1, \tau_2\})| \sim C\epsilon(\tau_2 - \tau_1)^{-(1+\alpha)}$. This analysis (proof of the Kotecky-Preiss criterion) is accomplished in Section 3.1.

In the concluding Section 4, we prove our results. The main point of this section is to pinpoint how the cluster expansion gives rise to decay of correlations. Such reasoning is completely standard in high-temperature expansions of statistical physics, see e.g. [27].

To prove the photon bound, we use an analogous approach but this time we develop a polymer representation like (1.28) for the quantity $\mathrm{Tr}(\rho_t \mathrm{e}^{\kappa N})$. In fact, in our proofs, we provide one general polymer representation for $\mathrm{Tr}(O\mathrm{e}^{(\kappa/2)N}\rho_t \mathrm{e}^{(\kappa/2)N})$ and then we set $\kappa = 0$ to study $\mathrm{Tr}(\rho_t O)$ and $O = \mathbb{1}$ to study $\mathrm{Tr}(\rho_t \mathrm{e}^{\kappa N})$.

In the rest of the paper, we assume that Assumptions A, B and C are satisfied for $\alpha > 0$ and all the Lemmata will depend on this parameter α.

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2 Polymer Representation

In this section, we complete the first important step of our proof, namely we rewrite all quantities of interest through a polymer representation. First, let us discretize time by introducing a mesoscopic time scale λ^{-2}, where $\lambda > 0$ is the coupling strength. That is, we consider times of the form $t = n/\lambda^2$ with $n \in \mathbb{N}$ (the discretization will be easily removed at the end of the argument). The main quantity that we study is

$$Z_n = Z_n(O, \rho_0, \kappa) := \mathrm{Tr}\left[O\mathrm{e}^{(\kappa/2)N}\mathrm{e}^{-\mathrm{i}(n/\lambda^2)H}\rho_0 \mathrm{e}^{\mathrm{i}(n/\lambda^2)H}\mathrm{e}^{(\kappa/2)N}\right] \tag{2.1}$$

where N is the number operator on the Fock space $\mathscr{H}_{\mathrm{F}}$ and $\kappa \in \mathbb{C}$ is a sufficiently small parameter. The operators ρ_0 and O are the initial states, respectively observable constructed as in Section 1.3.

As announced, we develop a polymer representation for Z_n. In fact, we will first construct a polymer representation with *operator-valued polymer weights*. In this representation, the polymers correspond to deviations from Markovian behavior. This is described in Section 2.1. Bounds on the operator-valued polymer weights are stated in 2.2 but their proof is deferred to Section 3.

Then we define the 'true' (scalar) polymer representation for Z_n by adding the excitations of the Markovian approximation to the already defined operator-valued polymers. This is done in Section 2.3. Why this leads to scalar polymers will be explained there. Finally, we need to provide bounds on the scalar polymers in order to satisfy the Kotecky-Preiss criterion. These bounds follow from the bounds on the operator valued polymer weights and this is described in Section 2.4. In all of the following sections, we prefer to treat 'bulk' and 'boundary'- polymers separately, where the terms 'bulk' and 'boundary' refer to the time-dimension. The analysis of the bulk polymers is the more subtle piece of work but the treatment of the boundary polymers demands additional notation. Therefore we treat the former first, and then indicate the (in all cases quite minimal) changes necessary for the latter.

2.1 Operator valued polymer model

2.1.1 Definition of the deformed dynamics

Starting from (2.1), we would like to move the deformation parameter κ into the dynamics $\mathrm{e}^{-\mathrm{i}tH}$. Given an operator X, we introduce the left, right and two-sided multiplication, acting on operators ρ

$$\mathcal{L}(X)\rho := X\rho, \qquad \mathcal{R}(X)\rho := \rho X, \qquad \mathcal{M}(X)\rho := X\rho X. \tag{2.2}$$

Defining the Liouvillian $L = \mathrm{ad}(H) = \mathcal{L}(H) - \mathcal{R}(H)$ with H as in (1.5), we may rewrite (2.1) as

$$Z_n = \mathrm{Tr}\left[O\mathcal{M}(\mathrm{e}^{(\kappa/2)N})\mathrm{e}^{-\mathrm{i}(n/\lambda^2)L}\rho_0\right]. \tag{2.3}$$

We formally define the *deformed* (not self-adjoint) Hamiltonian

$$H_\kappa = \mathrm{e}^{(\kappa/2)N} H \mathrm{e}^{-(\kappa/2)N} \tag{2.4}$$

and the *deformed* Liouvillian

$$L_\kappa := \mathcal{L}(H_\kappa) - \mathcal{R}(H_{-\kappa}) = \mathcal{M}(\mathrm{e}^{(\kappa/2)N})\, L\, \mathcal{M}(\mathrm{e}^{-(\kappa/2)N}). \tag{2.5}$$

Then, formally

$$\mathcal{M}(\mathrm{e}^{(\kappa/2)N})\mathrm{e}^{-\mathrm{i}tL}\mathcal{M}(\mathrm{e}^{-(\kappa/2)N}) = \mathrm{e}^{-\mathrm{i}tL_\kappa} \tag{2.6}$$

which would allow to rewrite

$$Z_n = \mathrm{Tr}\left[O\mathrm{e}^{-\mathrm{i}(n/\lambda^2)L_\kappa}\mathcal{M}(\mathrm{e}^{(\kappa/2)N})\rho_0\right]. \tag{2.7}$$

To make these manipulations precise note first that relying on the relative boundedness of $\mathrm{e}^{(\kappa/2)N} H_\mathrm{I} \mathrm{e}^{-(\kappa/2)N}$ w.r.t. $H_\mathrm{S} + H_\mathrm{F}$, which follows from (1.6), it is easy to construct H_κ and L_κ as unbounded closed operators and to show that they form an analytic family (in κ) of class A in the sense of Kato [21]. Then (2.7) is a simple consequence of the functional calculus in case $\kappa \in \mathrm{i}\mathbb{R}$. The upcoming Lemma 2.1 gives a constructive meaning to it for arbitrary κ and provides an expansion that we will use in practice. To state this lemma, we need a few additional definitions. First, some additional Liouvillians:

$$L_\mathrm{S} := \mathrm{ad}(H_\mathrm{S}), \qquad L_\mathrm{F} := \mathrm{ad}(H_\mathrm{F}), \qquad L_\mathrm{I} := \lambda \mathrm{ad}(H_\mathrm{I}) \tag{2.8}$$

and

$$L_{\mathrm{I},\kappa}(s) := \mathcal{M}(\mathrm{e}^{(\kappa/2)N})\mathrm{e}^{\mathrm{i}s(L_\mathrm{S}+L_\mathrm{F})} L_\mathrm{I} \mathrm{e}^{-\mathrm{i}s(L_\mathrm{S}+L_\mathrm{F})}\mathcal{M}(\mathrm{e}^{-(\kappa/2)N}). \tag{2.9}$$

Defining

$$\begin{aligned} \Phi_\kappa(\psi, s) &:= \mathrm{e}^{(\kappa/2)N}\mathrm{e}^{\mathrm{i}s(H_\mathrm{S}+H_\mathrm{F})}\Phi(\psi)\mathrm{e}^{-\mathrm{i}s(H_\mathrm{S}+H_\mathrm{F})}\mathrm{e}^{-(\kappa/2)N} \\ &= \int \mathrm{d}q\, (\mathrm{e}^{\mathrm{i}s|q|+\kappa/2}\psi(q)a_q^* + \mathrm{e}^{-\mathrm{i}s|q|-\kappa/2}\overline{\psi(q)}a_q) \end{aligned} \tag{2.10}$$

we have

$$L_{\mathrm{I},\kappa}(s) = \lambda\mathcal{L}\left(\mathrm{e}^{\mathrm{i}sL_\mathrm{S}}(D)\otimes\Phi_\kappa(\phi,s)\right) - \mathcal{R}\left(\mathrm{e}^{\mathrm{i}sL_\mathrm{S}}(D)\otimes\Phi_{-\kappa}(\phi,s)\right). \tag{2.11}$$

For $\Psi \in \mathscr{H}$, we decompose $\Psi = \sum_{m\in\mathbb{N}}\Psi_m$ where $\Psi_m = 1_{[N=m]}\Psi$, i.e. $\Psi_m \in \mathscr{H}_\mathrm{S}\otimes\mathscr{H}_{\mathrm{F},m}$ with $\mathscr{H}_{\mathrm{F},m} \subset \mathscr{H}_\mathrm{F}$ the m-boson sector, see e.g. [9] for more details. Define the dense subspace $\mathscr{D}_1(\mathscr{H}) \subset \mathscr{B}_1(\mathscr{H})$ to be the space consisting of finite linear combinations of rank-one operators $|\Psi\rangle\langle\Psi'|$ satisfying

$$\forall m \in \mathbb{N} : \max\left(\|\Psi_m\|, \|\Psi'_m\|\right) \leq \frac{C^m}{\sqrt{m!}}, \qquad \text{for some } C > 0 \tag{2.12}$$

Lemma 2.1. *The LHS of equation* (2.6) *defines an operator*

$$\mathrm{e}^{-\mathrm{i}tL_\kappa} : \mathscr{D}_1(\mathscr{H}) \to \mathscr{D}_1(\mathscr{H})$$

for all $\kappa \in \mathbb{C}$ and $t \in \mathbb{R}$. Given $\rho \in \mathscr{D}_1(\mathscr{H})$ the map $\kappa \mapsto \mathrm{e}^{-\mathrm{i}tL_\kappa}\rho$ is holomorphic from $\mathbb{C}$ to $\mathscr{B}_1(\mathscr{H}_\mathrm{S})$. Moreover on this domain

$$\mathrm{e}^{-\mathrm{i}(t_1+t_2)L_\kappa} = \mathrm{e}^{-\mathrm{i}t_1 L_\kappa}\mathrm{e}^{-\mathrm{i}t_2 L_\kappa} \tag{2.13}$$

and

$$\mathrm{e}^{-\mathrm{i}tL_\kappa}\rho = \mathrm{e}^{-\mathrm{i}t(L_\mathrm{S}+L_\mathrm{F})}\sum_{m\in\mathbb{N}}(-\mathrm{i})^m \int\limits_{0<t_1<\ldots<t_m<t} \mathrm{d}t_1\ldots\mathrm{d}t_m\, L_{\mathrm{I},\kappa}(t_m)\ldots L_{\mathrm{I},\kappa}(t_2)L_{\mathrm{I},\kappa}(t_1)\rho \tag{2.14}$$

where the $m = 0$-term on the RHS is understood to equal ρ and sums and integrals converge absolutely. Finally, the RHS of eq. (2.7) *is well defined and* (2.7) *holds.*

Proof. Starting from $L_\kappa = L_{\rm S} + L_{\rm F} + \lambda L_{{\rm I},\kappa}$, and iterating the Duhamel formula

$$\mathrm{e}^{-\mathrm{i}tL_\kappa}\rho = \mathrm{e}^{-\mathrm{i}t(L_{\rm F}+L_{\rm S})}\rho - \mathrm{i}\int_0^t \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}(t-s)L_\kappa} L_{{\rm I},\kappa}\mathrm{e}^{-\mathrm{i}s(L_{\rm F}+L_{\rm S})}\rho \tag{2.15}$$

we formally arrive at (2.14). Hence, the only nontrivial claim in the lemma is the absolute convergence of the series on the RHS of (2.14) and the fact that it belongs to $\mathscr{D}_1(\mathscr{H})$. We refer to [10] for an explicit proof, which relies exclusively on the well-known estimate

$$\left\| \int \mathrm{d}q \psi(q) a^\#_q \Psi_m \right\|_{\mathscr{H}} \leq \sqrt{m+1}\, \|\Psi_m\|_{\mathscr{H}} \int \mathrm{d}q |\psi(q)|^2, \qquad \text{for } a^\# = a, a^* \tag{2.16}$$

□

Hence, in what follows, we freely use the operators $\mathrm{e}^{-\mathrm{i}tL_\kappa}$ and the group property (2.13).

2.1.2 Splitting of the dynamics

We define the reduced dynamics $Q_t : \mathscr{B}_1(\mathscr{H}_{\rm S}) \to \mathscr{B}_1(\mathscr{H}_{\rm S})$ of the atom;

$$Q_t\rho_{\rm S} := \mathrm{Tr}_{\rm F}[\mathrm{e}^{-\mathrm{i}tL_\kappa}(\rho_{\rm S}\otimes P_\Omega)] \tag{2.17}$$

where $\mathrm{Tr}_{\rm F} : \mathcal{B}_1(\mathscr{H}) \to \mathcal{B}_1(\mathscr{H}_{\rm S})$ is the partial trace and the well-definedness of the RHS follows from Lemma 2.1. A large part of our analysis serves to prove that Q_t tends to a one-dimensional projection as $t \to \infty$.

We start by rewriting Z_n in (2.7). Recall $L_{\rm F} = \mathrm{ad}(H_{\rm F})$ and introduce operators U_τ with $\tau \in \mathbb{N}$.

$$U_\tau \;\; = \;\; \mathrm{e}^{\mathrm{i}(\tau/\lambda^2)L_{\rm F}}\mathrm{e}^{-\mathrm{i}(1/\lambda^2)L_\kappa}\mathrm{e}^{-\mathrm{i}((\tau-1)/\lambda^2)L_{\rm F}} \tag{2.18}$$

The motivation for this definition is that the product of U_τ telescopes into

$$U_n \ldots U_1 = \mathrm{e}^{\mathrm{i}(n/\lambda^2)L_{\rm F}}\mathrm{e}^{-\mathrm{i}(n/\lambda^2)L_\kappa}. \tag{2.19}$$

In particular, if we choose $\rho_0 = \rho_{{\rm S},0}\otimes P_\Omega$ and $O = O_{\rm S}\otimes \mathbb{1}$ then eq. (2.7) becomes

$$Z_n = \mathrm{Tr}_{\rm S}[O_{\rm S} Q_{n/\lambda^2}\rho_{{\rm S},0}] \tag{2.20}$$

and hence in this case the study of Z_n reduces to the study of Q_t. The main idea of our approach is that, at least qualitatively, the main contribution to Q_{n/λ^2} can be inferred by approximating Q_{n/λ^2} by $(Q_{1/\lambda^2})^n$. We rename $T := Q_{1/\lambda^2}$ and we define the 'excitation operators'

$$B_\tau = U_\tau - T\otimes\mathbb{1} \tag{2.21}$$

Our task is to understand how the behavior of T^n is modified by the excitation operators B_τ. To quantify the influence of the latter, we now develop a formalism.

2.1.3 Correlation functions of excitations

We abbreviate

$$\mathscr{R}_{\rm S} = \mathscr{B}(\mathscr{B}_1(\mathscr{H}_S)), \qquad \mathscr{R}_{\rm F} = \mathscr{B}(\mathscr{B}_1(\mathscr{H}_{\rm F})) \tag{2.22}$$

Define, for $W, W' \in \mathscr{R}_{\rm S}\otimes\mathscr{R}_{\rm F}$ the object

$$W \otimes_{\rm S} W' \in \mathscr{R}_{\rm S}\otimes\mathscr{R}_{\rm S}\otimes\mathscr{R}_{\rm F}$$

as an operator product in F-part and tensor product in S-part. Concretely, let $W = W_{\rm S}\otimes W_{\rm F}$ and $W' = W'_{\rm S}\otimes W'_{\rm F}$. Then

$$W\otimes_{\rm S} W' := W_{\rm S}\otimes W'_{\rm S}\otimes W_{\rm F}W'_{\rm F}.$$

and we extend this by linearity to arbitrary W, W'. Iterating this construction we define for $W_i \in \mathscr{R}_{\rm S} \otimes \mathscr{R}_{\rm F}$, $i = 1, \ldots, m$

$$W_m \otimes_{\rm S} \ldots\ldots \otimes_{\rm S} W_2 \otimes_{\rm S} W_1 \in (\mathscr{R}_S)^{\otimes^m} \otimes \mathscr{R}_{\rm F}.$$

(Note that no analysis problems arise since $\mathscr{R}_{\rm S}$ is finite-dimensional.) We define the 'expectation'

$$\mathbb{E} : (\mathscr{R}_{\rm S})^{\otimes^m} \otimes \mathscr{R}_{\rm F} \;\to\; (\mathscr{R}_{\rm S})^{\otimes^m}$$

as

$$\mathbb{E}(W)J := {\rm Tr}_{\rm F}[W(J \otimes P_\Omega)], \qquad J \in (\mathscr{B}_1(\mathscr{H}_{\rm S}))^{\otimes^m}$$

Obviously, the action of $\mathbb{E}$ is extended to unbounded W satisfying $W((\mathscr{B}_1(\mathscr{H}_{\rm S}))^{\otimes^m} \otimes P_\Omega) \in \mathscr{B}_1(\mathscr{H}_{\rm S}^{\otimes^m} \otimes \mathscr{H}_{\rm F})$. In particular, by (an obvious generalization of) Lemma 2.1, we can consider $W = U_{\tau_m} \otimes_{\rm S} \ldots \otimes_{\rm S} U_{\tau_1}$ for any m-tuple of times $\tau_1, \ldots, \tau_m$. For example, we rewrite the definition of T as $T = \mathbb{E}(U_\tau)$ (this is with $m = 1$). Let $A = \{\tau_1, \tau_2, \ldots, \tau_m\} \subset \mathbb{N}$ with the convention that $\tau_i < \tau_{i+1}$ and define the 'time-ordered correlation function'

$$G_A := \mathbb{E}\left(B_{\tau_m} \otimes_{\rm S} B_{\tau_{n-1}} \otimes_{\rm S} \cdots \otimes_{\rm S} B_{\tau_1}\right) \in (\mathscr{R}_S)^{\otimes^m} \tag{2.23}$$

Note that $G_A = 0$ when the set A is a singleton, as follows directly from $B_\tau = U_\tau - \mathbb{E}(U_\tau) \Rightarrow \mathbb{E}(B_\tau) = 0$, and $G_A = G_{A+\tau}$ because ${\rm e}^{-{\rm i}tL_{\rm F}} P_\Omega = P_\Omega$.

It will be convenient to label the $\mathscr{R}_{\rm S}$'s and to drop the subscript S (since we will rarely need $\mathscr{R}_{\rm F}$). Therefore, let $\mathscr{R}_\tau$, $\tau \in \mathbb{N}$ be copies of $\mathscr{R}_{\rm S}$ and let $A \subset \mathbb{N}$ be as above. We define $\mathscr{R}_A$ by

$$\mathscr{R}_A := \mathscr{R}_{\tau_m} \otimes \mathscr{R}_{\tau_{m-1}} \otimes \cdots \otimes \mathscr{R}_{\tau_1}.$$

Obviously, $\mathscr{R}_A$ is naturally isomorphic to $\mathscr{R}^{\otimes^m}$ by identifying the right-most tensor factor to $\mathscr{R}_1$, the next one to $\mathscr{R}_2$, etc. We denote this isomorphism from $\mathscr{R}^{\otimes^m}$ to $\mathscr{R}_A$ by $\boldsymbol{I}_A$ and we will from now on write G_A to denote $\boldsymbol{I}_A[G_A]$ since G_A acting on the unlabeled space $\mathscr{R}^{\otimes^m}$ will not be used. Similarly we will most often abbreviate $\boldsymbol{I}_\tau[V]$ ($V \in \mathscr{R}$) by V_τ, which will lead to a slight abuse of notation, see below and in Section 2.1.6. Consider a collection $\mathcal{A}$ of disjoint sets A , then each of the spaces $\mathscr{R}_{A\in\mathcal{A}}$ is naturally embedded into $\mathscr{R}_{{\rm Supp}\mathcal{A}}$, where ${\rm Supp}\mathcal{A} = \cup_{A\in\mathcal{A}} A$. Consequently, given a collection of operators $K_A \in \mathscr{R}_A$, $A \in \mathcal{A}$, we can define

$$\otimes_{A\in\mathcal{A}}\, K_A \in \mathscr{R}_{{\rm Supp}\mathcal{A}} \tag{2.24}$$

In particular, we have $\mathscr{R}_A = \otimes_{\tau\in A}\mathscr{R}_\tau$. In the literature on quantum lattice systems, where similar constructions are necessary, one usually identifies $\mathscr{R}_\tau$ with the subspace $\ldots \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathscr{R} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \ldots$ of the infinite tensor product, such that, for example, $\otimes_{A\in\mathcal{A}} K_A$ is simply written as $\prod_A K_A$. However, we chose to have the tensor products explicit in the notation.

We define the "contraction operator" $\mathcal{T} : \mathscr{R}_A \to \mathscr{R}$, by first giving its action on elementary tensors. Consider a family of operators $V_\tau \in \mathscr{R}$, and set

$$\mathcal{T}\left[\bigotimes_{\tau\in A} V_\tau\right] = V_{\tau_m} V_{\tau_{m-1}} \ldots V_{\tau_1}, \qquad \text{where} \qquad \tau_m > \tau_{m-1} > \ldots > \tau_1 \tag{2.25}$$

and then extend linearly to the whole of $\mathscr{R}_A$. On the LHS, we abbreviated $\boldsymbol{I}_\tau[V_\tau]$ by V_τ.

By expanding $U_\tau = T \otimes 1 + B_\tau$ for every $\tau \in I_n := \{1, \ldots, n\}$ in the expression for the reduced dynamics (2.17), we arrive at

$$Q_{n/\lambda^2} = \sum_{A\subset I_n} \mathcal{T}\left[G_A \bigotimes_{\tau\in I_n\setminus A} T_\tau\right]. \tag{2.26}$$

Note that, for each operator appearing in the tensor product, we have specified the space $\mathscr{R}_A$ on which it acts. In contrast, the order in which we write the factors inside the $\mathcal{T}[\cdot]$ does not have any meaning.

Remark 2.2. *The above construction with tensor product spaces $\mathscr{R}_A$ and the contraction operator $\mathcal{T}$ does of course not depend on the fact that the spaces are indexed by elements of $\mathbb{N}$. The only requirement is that the index set is ordered. In particular, in Section 3.1, we will use the same formalism, but now with copies of $\mathscr{R}$ indexed by (a finite number of) times $t_i \in \mathbb{R}^+$.*

2.1.4 Polymer expansion for cumulants

The cumulants or "connected correlation functions", denoted by G^c_A, are defined to be operators in $\mathscr{R}_A$ satisfying $G^c_{\{\tau\}} = G_{\{\tau\}} = 0$ and

$$G_{A'} = \sum_{\text{partitions } \mathcal{A} \text{ of } A'} \left(\bigotimes_{A\in\mathcal{A}} G^c_A \right) \tag{2.27}$$

The tensor product in this formula makes sense since $\mathscr{R}_{A'} = \otimes_{A\in\mathcal{A}}\mathscr{R}_A$ whenever $\mathcal{A}$ is a partition of A' (cfr. previous section). Note that this definition of connected correlation functions reduces to the usual probabilistic definition when all operators that appear are numbers and the tensor product can be replaced by multiplication. Just as in the probabilistic case, the relations (2.27) for all sets A' fix the operators G^c_A uniquely since the formula (2.27) can be inverted.

Plugging (2.27) into (2.26), we get

$$Q_{n/\lambda^2} = \sum_{\mathcal{A}\in\mathfrak{B}^0_n} \mathcal{T}\left[\bigotimes_{A\in\mathcal{A}} G^c_A \bigotimes_{\tau\in I_n\setminus \mathrm{Supp}\mathcal{A}} T_\tau \right] \tag{2.28}$$

where $\mathfrak{B}^0_n$ is the set of disjoint collections of subsets of I_n. The formula (2.28) is the starting point of our analysis. Note that in the case where $\mathcal{A}$ contains at least one set A that is not a discrete interval (a consecutive set of integers), there is no obvious way to write the RHS of (2.28) as an operator product. This was the main motivation for introducing the formalism with tensor products and the contraction $\mathcal{T}$. Following a standard terminology in statistical mechanics we call the sets $A \subset I_n$ *polymers* and the function G^c_A *weight* of the polymer. Unlike in statistical mechanics this weight is operator-valued and our objective is to manipulate this expansion to arrive to standard scalar valued weights.

2.1.5 Notation for combinatorics

We gather some notation (partially already introduced above) that will be used throughout. We let $I_n = \{1,\dots,n\}$ which is interpreted as the set of discrete times, often denoted by $\tau,\tau'\dots$. It will be convenient to add two "boundary elements" to this set, which are represented by the times 0 and $n+1$, hence we set

$$\breve{I}_n = I_n \cup \{0, n+1\} \tag{2.29}$$

The set $\mathfrak{B}_n$ is the set of collections of subsets of I_n, i.e. $\mathfrak{B}_n = 2^{(2^{I_n})}$. Elements of $\mathfrak{B}_n$ will mostly be denoted by $\mathcal{A} = \{A_1, A_2, \dots, A_m\}$ with $A_i \subset I_n$. We define two important subsets of $\mathfrak{B}_n$: $\mathfrak{B}^0_n$ is the set of collections of mutually disjoint sets in I_n, and $\mathfrak{B}^1_n$ is the set of collections of sets such that the distance between any of them is greater than 1;

$$\mathfrak{B}^j_n := \{\mathcal{A}\subset 2^{I_n} \,\big|\, \forall A, A'\in\mathcal{A} : (A\neq A' \Rightarrow \mathrm{dist}(A,A') > j)\}, \qquad j = 0,1 \tag{2.30}$$

where $\mathrm{dist}(A,A') = \min_{\tau\in A,\tau'\in A'}|\tau-\tau'|$. Similarly, we define $\breve{\mathfrak{B}}_n, \breve{\mathfrak{B}}^0_n, \breve{\mathfrak{B}}^1_n$ starting from $\breve{I}_n$ instead of I_n. Intervals in I_n and $\breve{I}_n$ are sets of consecutive numbers. For any A, we say that the intervals $J_1\dots,J_m$ are the maximal intervals in A iff. $\cup_j J_j = A$ and the collection $\{J_1,\dots,J_m\}$ is in $\mathfrak{B}^1_n$ or $\breve{\mathfrak{B}}^1_n$.

We define in general the support of $\mathcal{A}\in\breve{\mathfrak{B}}_n$ as

$$\mathrm{Supp}\mathcal{A} = \cup_{A\in\mathcal{A}} A \tag{2.31}$$

and the 'span' of sets as

$$\mathrm{Span}A := \{\min A,\dots,\max A\}, \qquad \mathrm{Span}\mathcal{A} = \mathrm{Span}(\mathrm{Supp}\mathcal{A}) \tag{2.32}$$

that is $\mathrm{Span}A, \mathrm{Span}\mathcal{A}$ is the smallest interval that contains $A, \mathrm{Supp}\mathcal{A}$, respectively. The size of $\mathrm{Span}A$, $\mathrm{Supp}\mathcal{A}$ is called the diameter of A, $\mathcal{A}$, denoted by

$$d(A) := 1 + \max A - \min A, \qquad d(\mathcal{A}) = 1 + \max \mathrm{Supp}\mathcal{A} - \min \mathrm{Supp}\mathcal{A} \tag{2.33}$$

2.1.6 Initial state and observable

To deal with the observable O and initial state ρ_0 in a convenient way we also define the operators U_0, U_{n+1}

$$U_0\rho := \mathrm{e}^{(\kappa/2)N}\mathcal{W}(\psi_\ltimes)\rho\mathcal{W}^*(\psi_\ltimes)\mathrm{e}^{(\kappa/2)N}, \tag{2.34}$$

$$U_{n+1}\rho := \mathcal{W}(\mathrm{e}^{\mathrm{i}(n/\lambda^2)|q|}\psi_\rtimes)\rho \tag{2.35}$$

where we wrote $\mathcal{W}(\psi)$ instead of $\mathbb{1}\otimes\mathcal{W}(\psi)$. Note that U_{n+1} depends on the total macroscopic time n, which is a notational drawback of our formalism. We introduce a modified reduced dynamics $\breve{Q}_n$, as

$$\breve{Q}_n\rho_{\mathrm{S}} := \mathrm{Tr}_{\mathrm{F}}\left[U_{n+1}\mathrm{e}^{\mathrm{i}(n/\lambda^2)L_{\mathrm{F}}}\mathrm{e}^{-\mathrm{i}(n/\lambda^2)L_\kappa}U_0(\rho_{\mathrm{S}}\otimes P_\Omega)\right] \tag{2.36}$$

such that we have

$$Z_n(O,\rho_0,\kappa) = \mathrm{Tr}_{\mathrm{S}}[O_{\mathrm{S}}\breve{Q}_n\rho_{\mathrm{S},0}] \tag{2.37}$$

which generalizes (2.20) and reduces to the latter when $\psi_\ltimes = \psi_\rtimes = 0$. To check (2.37), note that $\mathcal{W}(\mathrm{e}^{\mathrm{i}t|q|}\psi_\rtimes) = \mathrm{e}^{\mathrm{i}tH_{\mathrm{F}}}\mathcal{W}(\psi_\rtimes)\mathrm{e}^{-\mathrm{i}tH_{\mathrm{F}}}$.

It is straightforward to extend the formalism of Sections 2.1.3-2.1.4 to incorporate the times $0, n+1$. We define

$$T_0 := \mathbb{E}(U_0) = \left\langle \mathrm{e}^{(\kappa/2)N}\mathcal{W}(\psi_\ltimes)\Omega, \mathrm{e}^{(\kappa/2)N}\mathcal{W}(\psi_\ltimes)\Omega\right\rangle \mathbb{1} \tag{2.38}$$

$$T_{n+1} := \mathbb{E}(U_{n+1}) = \langle\Omega, \mathcal{W}(\psi_\rtimes)\Omega\rangle\,\mathbb{1} \tag{2.39}$$

and set

$$B_\tau := U_\tau - T_\tau, \qquad \tau = 0, n+1. \tag{2.40}$$

Next, we define copies $\mathscr{R}_0, \mathscr{R}_{n+1}$ of $\mathscr{R}$, and also $\mathscr{R}_A$ for $A \cap \{0, n+1\} \neq \emptyset$, completely analogous to the construction in Section 2.1.3, such that the definition of (connected) correlation functions G^{c}_A extends to all $A \subset \breve{I}_n$. In our expansion, we will need $\boldsymbol{I}_0[T_0], \boldsymbol{I}_{n+1}[T_{n+1}]$ which we write simply as T_0, T_{n+1}. With these definitions, the representation (2.28) is generalized to

$$\breve{Q}_n = \sum_{\mathcal{A}\in\breve{\mathfrak{B}}^0_n} \mathcal{T}\left[\bigotimes_{A\in\mathcal{A}} G^{\mathrm{c}}_A \bigotimes_{\tau\in\breve{I}_n\setminus\mathrm{Supp}\mathcal{A}} T_\tau\right] \tag{2.41}$$

2.1.7 Norms

Let us introduce some conventions. For S acting on $\mathscr{H}_{\mathrm{S}}$, we write

$$\|S\| := \sup_{\psi\in\mathscr{H}_{\mathrm{S}},\|\psi\|=1} \|S\psi\| \tag{2.42}$$

and for E acting on $\mathscr{B}(\mathscr{H}_{\mathrm{S}})$, we write

$$\|E\| := \sup_{\rho\in\mathscr{B}_1(\mathscr{H}_{\mathrm{S}}),\|\rho\|_1=1} \|E(\rho)\|_1, \qquad \text{with } \|\rho\|_1 = \mathrm{Tr}\,|\rho|, \tag{2.43}$$

i.e. the natural operator norm on $\mathscr{B}(\mathscr{B}_1(\mathscr{H}_{\mathrm{S}}))$.

For $E \in \mathscr{R}_A, |A| > 1$, we exploit that E can be written as a finite sum of elementary tensors , i.e.

$$E = \sum_\nu E_\nu, \qquad E_\nu = \otimes_{\tau\in A}E_{\nu,\tau}, \qquad E_{\nu,\tau} \in \mathscr{R}_\tau, \tag{2.44}$$

to define

$$\|E\|_\diamond := \inf_{\{E_\nu\}} \sum_\nu \prod_{\tau\in A} \|E_{\nu,\tau}\| \tag{2.45}$$

where the infimum ranges over all such elementary tensor-representations of E. This norm is useful because of the following properties (trivial from the definition):

1) For any family of operators $K_{A\in\mathcal{A}}$ with $K_A \in \mathscr{R}_A$ and $\mathcal{A}$ a collection of disjoint sets, i.e. $\mathcal{A} \in \breve{\mathfrak{B}}_n^0$, we have

$$\left\| \bigotimes_{A\in\mathcal{A}} K_A \right\|_\diamond \leq \prod_{A\in\mathcal{A}} \|K_A\|_\diamond \tag{2.46}$$

2) For any $K_A \in \mathscr{R}_A$,

$$\|\mathcal{T}[K_A]\|_\diamond \leq \|K_A\|_\diamond \tag{2.47}$$

2.2 Bounds on operator-valued polymers

We aim to set up a perturbative scheme where the G_A^c will describe small corrections to T, and hence we must specify in what sense the operators G_A^c are small.

We will often need to distinguish between *bulk* polymers, i.e. subsets of I_n, and boundary polymers, i.e. subsets $A \subset \breve{I}_n$ for which $A \cap \{0, n+1\} \neq \emptyset$. Lemma 2.3 gathers the necessary properties of operator valued polymers. To relate these properties to assumptions on the original model, we introduce the 'renormalized coupling constants'

$$\epsilon := |\lambda|^{2\min(\alpha,1)}, \qquad \breve{\epsilon} := \max(|\lambda|, \epsilon) \tag{2.48}$$

We use in general C, c to denote constants that can depend on all model parameters except the coupling constant λ, conjugation parameter κ, the macroscopic time n and the initial state and observable. By $\breve{C}, \breve{c}$ we denote constants that can also depend on the initial state and the observable (but not on λ, κ or the macroscopic time n). For operators $J, J' \in \mathscr{B}(\mathscr{H}_\mathrm{S})$, we use the Hilbert-Schmidt scalar product $\langle J, J'\rangle = \mathrm{Tr}[J^* J']$. Recall the diameter of A, $d(A)$, and write $d(A)^\alpha := (d(A))^\alpha$.

Lemma 2.3. *For sufficiently small $|\lambda|, |\kappa|$, but $\lambda \neq 0$, the following hold uniformly in λ, κ (i.e. g_T and all constants can be chosen independent of λ, κ)*

1) *The operator T has a simple eigenvalue e^{θ_T} with $\theta_T = \theta_T(\lambda, \kappa)$ and a gap $\mathrm{g}_T > 0$, in the sense that*

$$\|T^m - \mathrm{e}^{m\theta_T} R\| \leq C_T \mathrm{e}^{m(|\theta_T| - \mathrm{g}_T)}, \qquad m \in \mathbb{N} \tag{2.49}$$

where $R = R(\lambda, \kappa)$ is the (one-dimensional) spectral projector associated to the eigenvalue e^{θ_T}. Moreover, $\theta_T(\lambda, \kappa = 0) = 0$ and $\lim_{\lambda\to 0} \theta_T(\lambda, \kappa) = 0$. R is of the form

$$R = |\tilde{\eta}\rangle\langle\eta|. \tag{2.50}$$

where, for $\kappa = 0$, we can choose $\tilde{\eta} = 1$ and η a density matrix. For $\kappa \in \mathbb{R}$, we can choose $\eta, \tilde{\eta}$ positive.

2) *Let $\epsilon, \breve{\epsilon}$ be as in* (2.48)*. The bulk polymers satisfy*

$$\max_{\tau\in I_n} \sum_{A\subset I_n : A\ni\tau} (C\epsilon)^{-(|A|-1)} d(A)^\alpha \|G_A^c\|_\diamond \leq 1 \tag{2.51}$$

For boundary polymers, take $\tau = 0, n+1$, then,

$$\sum_{A\subset \breve{I}_n : A\ni\tau, A\neq\{0,n+1\}} (C\breve{\epsilon})^{-|A\cap I_n|} d(A)^\alpha \|G_A^c\|_\diamond \leq \breve{C} \tag{2.52}$$

For $A = \{0, n+1\}$ (excluded from the sum above), we have $d(A)^\alpha \|G_A^c\|_\diamond \leq \breve{C}$.

The proof of this lemma is in Section 3.

2.3 Scalar polymers

The representation (2.28) evokes the picture of a leading dynamics T interrupted by excitations, indexed by the sets $A \in \mathcal{A}$. One could call this representation a polymer expansion, but it is not yet what we want because the values of the polymers, i.e. G_A^c, are operators. To make them scalar, we exploit the dissipativity of the model,

namely the fact (see Lemma 2.3) that the reduced dynamics T has a maximal (in modulus), simple eigenvalue. Recalling that the corresponding spectral projection is denoted by R we write

$$T = RT + R_\perp T, \qquad RT = \mathrm{e}^{\theta_T} R \tag{2.53}$$

where $R_\perp := 1 - R$ and we have $RT = TR$. Hence, in particular

$$RTR_\perp T = R_\perp TRT = 0. \tag{2.54}$$

We insert this decomposition into the expansion (2.28): Let $\mathcal{J}(\mathcal{A}) = \{J_1, \dots, J_k\}$ be the family of maximal intervals in $I_n \setminus \mathrm{Supp}\mathcal{A}$. The relation (2.54) implies that each tensor product $\bigotimes_{\tau \in J_j} T_\tau$ in (2.28) may be replaced by $\bigotimes_{\tau \in J_j} (RT)_\tau + \bigotimes_{\tau \in J_j} (R_\perp T)_\tau$. Thus (2.28) becomes

$$Q_{n/\lambda^2} = \sum_{\mathcal{A} \in \mathfrak{B}_n^0} \sum_{\mathcal{J} \subset \mathcal{J}(\mathcal{A})} \mathcal{T} \left[\bigotimes_{A \in \mathcal{A}} G_A^c \bigotimes_{\tau \in \mathrm{Supp}\mathcal{J}} (R_\perp T)_\tau \bigotimes_{\tau \in I_n \setminus (\mathrm{Supp}\mathcal{J} \cup \mathrm{Supp}\mathcal{A})} (RT)_\tau \right]. \tag{2.55}$$

We will now produce a new family of polymers $\mathcal{A}' = \mathcal{A}'(\mathcal{A}, J)$ by "fusing" some of the sets in the family $\mathcal{A} \cup \mathcal{J}$. Let $\Gamma = \Gamma(\mathcal{A}, \mathcal{J})$ be the graph with vertex set $\mathcal{V}(\Gamma) = \mathcal{A} \cup \mathcal{J}$ and edges $\{S, S'\}$ whenever the sets S and S' are adjacent i.e. $\mathrm{dist}(S, S') = 1$ (in particular this implies that at least one of them is in $\mathcal{A}$). For each connected component γ of Γ write $\mathcal{V}(\gamma) = \mathcal{A}_\gamma \cup \mathcal{J}_\gamma$, set $A'_\gamma := \mathrm{Supp}\mathcal{A}_\gamma \cup \mathrm{Supp}\mathcal{J}_\gamma$ and let $\mathcal{A}'$ be the family of A'_γ. Defining

$$V((\mathcal{A}_\gamma, \mathcal{J}_\gamma)) := \bigotimes_{A \in \mathcal{A}_\gamma} G_A^c \bigotimes_{\tau \in \mathrm{Supp}\mathcal{J}_\gamma} (R_\perp T)_\tau, \tag{2.56}$$

the identity (2.55) becomes

$$Q_{n/\lambda^2} = \sum_{\mathcal{A} \in \mathfrak{B}_n^0} \sum_{\mathcal{J} \subset \mathcal{J}(\mathcal{A})} \mathcal{T} \left[\bigotimes_\gamma V((\mathcal{A}_\gamma, \mathcal{J}_\gamma)) \bigotimes_{\tau \in I_n \setminus \mathrm{Supp}\mathcal{A}'(\mathcal{A}, \mathcal{J})} (RT)_\tau \right]. \tag{2.57}$$

The next step is to write (2.57) as a sum of families $\mathcal{A}'$, i.e. to fix $\mathcal{A}'$ and sum over $\mathcal{A}$ and $\mathcal{J}$.

Definition 2.1. Let $\mathcal{A}$ be a family of mutually disjoint subsets of I_n and let $\mathcal{J}$ be a family of mutually disjoint and non-adjacent intervals in I_n and disjoint from the sets in $\mathcal{A}$, i.e. $\mathrm{Supp}\mathcal{A} \cap \mathrm{Supp}\mathcal{J} = \emptyset$. We say that $\mathcal{S} = (\mathcal{A}, \mathcal{J})$ is a *fusion* iff.

1) $\mathrm{dist}(I_n \setminus \mathrm{Supp}\mathcal{S}, \mathrm{Supp}\mathcal{J}) > 1$ where $\mathrm{Supp}\mathcal{S} := \mathrm{Supp}\mathcal{A} \cup \mathrm{Supp}\mathcal{J}$. (Pictorially; the intervals J are in the "interior" of $\mathrm{Supp}\mathcal{S}$.)
2) The graph $\Gamma(\mathcal{A}, \mathcal{J})$ is connected.

The set of fusions is denoted by $\mathfrak{S}_n^f$.

Defining

$$\Sigma V(A') := \sum_{\mathcal{S} = (\mathcal{A}, \mathcal{J}) \in \mathfrak{S}_n^f : \mathrm{Supp}\mathcal{S} = A'} V((\mathcal{A}, \mathcal{J})) \tag{2.58}$$

we obtain the representation for Q_{n/λ^2} in terms of fusions as (we drop the prime from A' and $\mathcal{A}'$)

$$Q_{n/\lambda^2} = \sum_{\mathcal{A} \in \mathfrak{B}_n^1} \mathcal{T} \left[\bigotimes_{\tau \in I_n \setminus \mathrm{Supp}\mathcal{A}} (RT)_\tau \bigotimes_{A \in \mathcal{A}} \Sigma V(A) \right]. \tag{2.59}$$

Note that by construction $\mathcal{A} \in \mathfrak{B}_n^1$ since the sets A'_γ above are non-adjacent i.e. their mutual distances are at least 2. Hence, for any $\mathcal{A}$ in the formula above, all τ that are adjacent to $\mathrm{Supp}\mathcal{A}$ carry the projector RT. A pictorial way to phrase this is that any $\mathcal{A}$ in (2.59) is surrounded by projections R. We exploit this by defining

$$\hat{v}(A') := \mathcal{T} \left[\Sigma V(A') \bigotimes_{\tau \in I_n \setminus A'} R_\tau \right], \qquad \hat{v}(A') \in \mathscr{R} \tag{2.60}$$

Note that $\hat{v}(A')$ is a multiple of R unless $A' \ni 1$ and/or $A' \ni n$. To eliminate these boundary effects, we consider $RQ_{n/\lambda^2}R$ instead of Q_{n/λ^2}, for the time being. Then (2.59) implies

$$RQ_{n/\lambda^2}R = \sum_{\mathcal{A}\in\mathfrak{B}_n^1} (RT)^{(n-|\mathrm{Supp}\mathcal{A}|)} \prod_{A\in\mathcal{A}} (R\hat{v}(A)R) \tag{2.61}$$

where the product on the RHS is commutative since the projection R is one-dimensional. It is convenient to extract the contribution due to the RT, by defining (recall $\eta, \tilde{\eta}$ from (2.50))

$$v(A) = \mathrm{e}^{-|A|\theta_T} \langle \tilde{\eta}, \hat{v}(A)\eta \rangle \tag{2.62}$$

Indeed, since $\langle \tilde{\eta}, \eta \rangle = 1$ (because $RR = R$), we get

$$RQ_{n/\lambda^2}R = Re^{n\theta_T} \sum_{\mathcal{A}\in\mathfrak{B}_n^1} \prod_{A\in\mathcal{A}} v(A) \tag{2.63}$$

and the RHS is indeed a 'scalar polymer representation', i.e. the weights $v(A)$ are numbers, and we can study it with the help of a cluster expansion. Note for later use that

$$\mathrm{Tr}\, RQ_{n/\lambda^2}R = Z_n(\tilde{\eta}\otimes\mathbb{1}, \eta\otimes P_\Omega, \kappa). \tag{2.64}$$

Our real object of interest however is the partition function (2.37) expressed in terms of the operator (2.41). We proceed with the latter as with Q_{n/λ^2}. Decompose as in (2.53)

$$T_\tau = RT_\tau + R_\perp T_\tau, \qquad \tau \in \{0, n+1\} \tag{2.65}$$

where R is the same operator as in (2.53). Since T_0, T_{n+1} are proportional to the identity operators, we have again $RT_\tau = T_\tau R$. As a consequence, for all $\tau, \tau' \in \breve{I}_n$

$$RT_\tau R_\perp T_{\tau'} = R_\perp T_\tau R T_{\tau'} = 0. \tag{2.66}$$

We end up with the expansion like (2.59) for $\breve{Q}_n$ with the expected difference that we have to replace I_n by $\breve{I}_n$ and $\mathfrak{B}_n^1$ by $\breve{\mathfrak{B}}_n^1$ (also in the definition of fusions).

For Z_n of (2.37) we get the scalar valued polymer expansion as in (2.63) with the change that the weights $v(A)$ for $A \cap \{0, n+1\} \neq \emptyset$ have to be slightly modified because the weight of the polymer is influenced by the initial state and the observable. We keep the definition of $\hat{v}(A)$ given in (2.60). Then, recall that $R = |\eta\rangle\langle\tilde{\eta}|$ and define

$$v(A) := \mathrm{e}^{-|A\cap I_n|\theta_T} k_A^{-1} \begin{cases} \langle \tilde{\eta}, \hat{v}(A)\rho_{\mathrm{S},0}\rangle & 0 \in A, n+1 \notin A \\ \langle O_{\mathrm{S}}, \hat{v}(A)\eta\rangle & n+1 \in A, 0 \notin A \\ \langle O_{\mathrm{S}}, \hat{v}(A)\rho_{\mathrm{S},0}\rangle & \{0, n+1\} \in A \end{cases} \tag{2.67}$$

where

$$k_A := \left(1_{[0\in A]} k_\ltimes + 1_{[0\notin A]}\right)\left(1_{[n+1\in A]} k_\rtimes + 1_{[n+1\notin A]}\right) \tag{2.68}$$

and

$$k_\ltimes := \langle \tilde{\eta}, T_0\rho_{\mathrm{S},0}\rangle = \mathrm{Tr}_{\mathrm{S}}(\tilde{\eta}, \rho_{\mathrm{S},0}) \langle \mathrm{e}^{(\kappa/2)N}\mathcal{W}(\psi_\ltimes)\Omega, \mathrm{e}^{(\kappa/2)N}\mathcal{W}(\psi_\ltimes)\Omega\rangle \tag{2.69}$$

$$k_\rtimes := \langle O_{\mathrm{S}}, T_{n+1}\eta\rangle = \mathrm{Tr}_{\mathrm{S}}(O_{\mathrm{S}}\eta)\, \mathrm{Tr}_{\mathrm{F}}(\mathcal{W}(\psi_\rtimes)P_\Omega) \tag{2.70}$$

With these definitions, we arrive at

$$Z_n = \mathrm{e}^{n\theta_T} k_\ltimes k_\rtimes \sum_{\mathcal{A}\in\breve{\mathfrak{B}}_n^1} \prod_{A\in\mathcal{A}} v(A) \tag{2.71}$$

which is our representation of Z_n as a polymer model. A few remarks:

1) The case where the factors $k_\ltimes, k_\rtimes$ vanish, will be of no concern, as we will explain later, following Lemma 4.2.

2) Up to now, we did not indicate the macroscopic time n in the notation. Let us, only in the next lines, indicate this dependence by a superscript. Consider A such that $\max A - 1 < n < n'$, then

$$v^{(n)}(A) = v^{(n')}(A). \tag{2.72}$$

Furthermore, $v^{(n)}(A+\tau) = v^{(n)}(A)$ for $\tau \in \mathbb{N}$, as long as $0 < \min A, \max A + \tau - 1 \leq n$. If we allow polymers to contain the final time $n+1$ but not the initial time 0, i.e. $0 < \min A, \max A - 1 \leq n < n'$, then we have

$$v^{(n)}(A) = v^{(n')}(A + (n' - n)). \tag{2.73}$$

These properties follow from the property $G_A = G_{A+\tau}$ and the expression for the 'final-time' operators $U_{n+1}, U_{n'+1}$.

3) If we choose $\rho_0 = \eta \otimes P_\Omega$, then $v(A) = 0$ whenever $0 \in A$. Indeed, Indeed, in that case we can substitute $\mathbb{1}$ for U_0, hence $B_0 = 0$, and we have $R_\perp \rho_{\mathrm{S},0} = 0$ because $\rho_{\mathrm{S},0} = \eta$. By analogous reasoning, we check that if we choose $O = \tilde{\eta} \otimes \mathbb{1}$, then $v(A) = 0$ whenever $n+1 \in A$. Note that these two observations are consistent with (2.63,2.64) which tell us that one can omit the boundary polymers.

We finish this section by a pictorial illustration of the construction. Each term in the representation of Q_{n/λ^2} in (2.28) can be represented by a picture as in (1). The horizontal axis is the time-axis which has been divided into intervals of the form $\lambda^{-2}[\tau - 1, \tau]$ by the discretization procedure. Each term in (2.28) is specified choosing a collection $\mathcal{A}$, this is indicated on our picture by connecting the τ that belong to the same $A \in \mathcal{A}$ by straight diagonal lines above the axis. The times $\tau \in \mathrm{Supp}\mathcal{A}$ are drawn with the symbol •—◆— and the $\tau \notin \mathrm{Supp}\mathcal{A}$ are drawn with a •⌒.

As to the operator value of this picture, each •⌒ corresponds to T, and each connected component of •—◆— corresponds to an operator G_A^c.

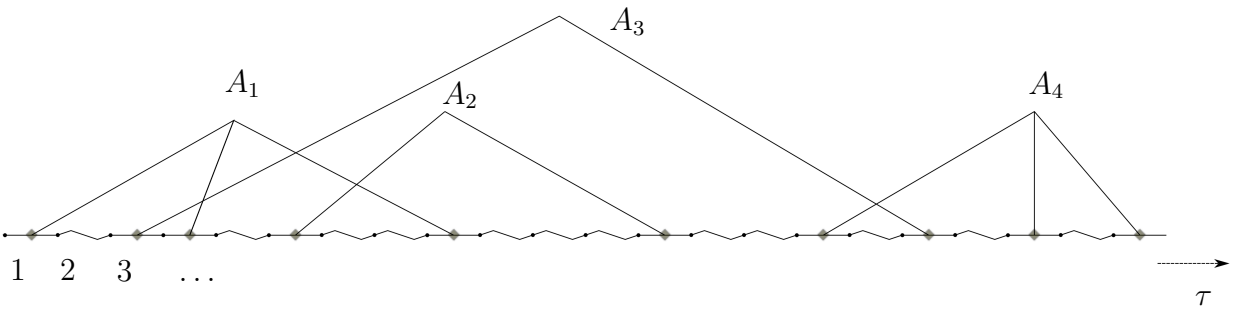

Figure 1: Representation of a term in the sum (2.28)

We now generate new pictures by splitting

$$T = TR + TR_\perp = \quad \bullet\!\!\frown\!\!\bullet \quad = \quad \sim\!\sim\!\sim \quad + \quad \Box\!\!\Box \tag{2.74}$$

In this way we obtain 2^ℓ pictures from Figure 1, with ℓ the number of •⌒. One of these is the upper picture in Figure 2. Note that splittings in which one ∿∿∿ is adjacent to an ▭▭ do not contribute because $RR_\perp = 0$. Therefore, we are in fact choosing the splitting for each maximal interval in $I_n \setminus \mathrm{Supp}\mathcal{A}$. In a next step, we do not distinguish between excitations that originate from B_τ or $TR_\perp$ and we write simply

▭ for both ▭▭ and •—◆—. Moreover, we fuse adjacent ▭ to form new polymers A', the lower picture in Figure 2 shows the result of fusing the upper picture. As a result, these new polymers are surrounded by ∿∿∿, and since those correspond to one-dimensional projectors, the operator-valued contribution of a new polymer to the total sum is independent of the presence of any other polymers. This is why the new representation is scalar.

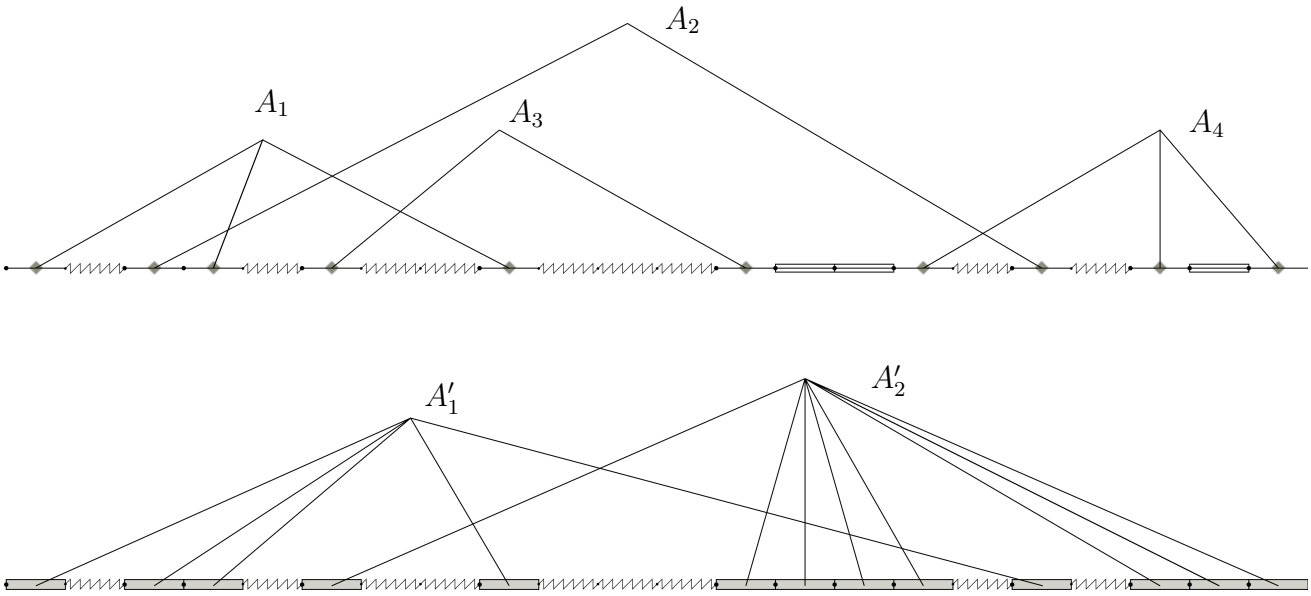

Figure 2: In the upper picture, we split $T = TR + TR_\perp$ for all $\tau \notin \cup_{i=1}^4 A_i$ and we choose one of the two terms for each τ. In the example displayed, we choose $TR_\perp$ for $\tau_{1,2,3} = 14, 15, 21$ (those that correspond to the triple lines in the upper picture), and TR for all other τ's. We fuse blocks marked with $TR_\perp$ with a polymer whenever they are adjacent to that polymer and we fuse two polymers whenever they are adjacent (possibly after having been fused with $TR_\perp$-blocks). In the example above, $A'_1 = A_1 \cup A_2$, $A'_2 = A_3 \cup A_4 \cup \{\tau_1, \tau_2, \tau_3\}$.

2.4 Bounds on scalar polymers

To obtain bounds on the new, scalar, polymers, we use the smallness of the operators G^c_A, expressed by the renormalized couplings $\epsilon, \breve{\epsilon}$, and their summability in A when one of the elements of A is held fixed. The $TR_\perp$ factors, which glue together several G^c_A-operators into one scalar polymer, are not small (they are of order 1) but whenever we line up ℓ of them, their weight decays exponentially in ℓ by the ergodicity of T (see Statement 1) of Lemma 2.3). Save for 2 special cases, dealt with in (2.77) below, all scalar polymers contain at least one operator G^c_A so that one can always extract a factor ϵ or $\breve{\epsilon}$.

Lemma 2.4 (bounds on polymer weights)**.** *For sufficiently small $|\lambda| \neq 0$, there is a $a_v > 0$ such that*

1) *For bulk polymers,*

$$\sum_{A \subset I_n : A \ni \tau} \mathrm{e}^{a_v |A|} d(A)^\alpha |v(A)| \leq C\epsilon, \tag{2.75}$$

2) *For boundary polymers, take $\tau = 0, n+1$, then*

$$\sum_{\substack{A \subset \breve{I}_n \,:\, A \ni \tau \\ A \neq \{0, n+1\},\, A \neq \breve{I}_n}} \mathrm{e}^{a_v |A|} d(A)^\alpha |v(A)| \leq \breve{C}\breve{\epsilon}, \tag{2.76}$$

and the boundary polymers that are excluded in the sum above satisfy

$$|v(\breve{I}_n)| \leq \breve{C}\mathrm{e}^{-2a_v n}, \qquad |v(\{0, n+1\})| \leq \breve{C} n^{-\alpha} \tag{2.77}$$

Proof. We first prove (2.75). Choose $A' \in I_n$. By the properties of the norm $\| \cdot \|_\diamond$ stated in Section 2.1.7 , we get

$$|v(A')| \;\leq\; C\mathrm{e}^{-|A| \mathrm{Re}\, \theta_T} \sum_{\mathcal{S} = (\mathcal{A}, \mathcal{J}) \in \mathfrak{S}^f_n : \mathrm{Supp} \mathcal{S} = A'} \|R\|^p_\diamond \prod_{A \in \mathcal{A}} \|G^c_A\|_\diamond \prod_{J \in \mathcal{J}} \|(TR_\perp)^{|J|}\|_\diamond \tag{2.78}$$

where p is the number of maximal intervals in $\mathrm{Span}A' \setminus A'$, and we will upper bound $p < C|\mathrm{Supp}\mathcal{A}|$. For any fusion $\mathcal{S} = (\mathcal{A}, \mathcal{J})$ such that $\mathrm{Supp}(\mathcal{A} \cup \mathcal{J}) = A'$, we have

$$d(A')^\alpha \leq \prod_{A\in\mathcal{A}} d(A)^\alpha \prod_{J\in\mathcal{J}} (1+|J|)^\alpha \tag{2.79}$$

which follows immediately from the connectedness of the graph $\Gamma(\mathcal{A}, \mathcal{J})$ in Definition 2.1.

To continue, let us interpret pairs $\mathcal{S} = (\mathcal{A}, \mathcal{J})$ as a collection of sets $S \in \mathcal{A}\cup\mathcal{J}$ together with a label indicating whether they are 'A-sets' ($\in \mathcal{A}$) or 'J-sets' ($\in \mathcal{J}$) (of course, only intervals can be J-sets). Let us gather such collections of labelled sets in the set $\mathfrak{S}_n$. Furthermore, we consider the adjacency relation $\sim$ defined by $S \sim S'$ iff. $d(S, S') = 1$ and at least one of the sets S, S' is an A-set. Then, it immediately follows that, for any fusion $\mathcal{S}$, the graph $(\mathcal{S}, \sim)$ is connected.

We introduce (with $C_w, c_w > 0$, to be fixed below)

$$w_\epsilon(S) := \begin{cases} d(A)^\alpha (\epsilon C_w)^{-(|A|-1)} \|G_A^c\|_\diamond & \text{if } S \text{ is an } A\text{-set} \\ C_w^{-1} \mathrm{e}^{-c_w|J|} & \text{if } S \text{ is a } J\text{-set} \end{cases} \tag{2.80}$$

Then we can bound the LHS of (2.75) as

$$\mathrm{e}^{a_v|A'|} d(A')^\alpha |v(A')| \leq \epsilon C \sum_{\mathcal{S}\in\mathfrak{S}_n^f : \mathrm{Supp}\mathcal{S} = A'} \prod_{S\in\mathcal{S}} w_\epsilon(S) \tag{2.81}$$

provided one chooses $c_w + a_v < \mathrm{g}_T$ and ϵ small enough (depending on C_w and c_w). To check (2.81), use (2.78, 2.79), the bounds $\|R\|_\diamond \leq C$ and $|\mathcal{J}| \leq C|\mathrm{Supp}\mathcal{A}|$ for any fusion $(\mathcal{A}, \mathcal{J})$, the fact that any fusion contains at least one A-set, and the bound from Lemma 2.3 1);

$$\mathrm{e}^{-|J|\mathrm{Re}\,\theta_T} \|(TR_\perp)^{|J|}\| \leq C_T \mathrm{e}^{-|J|\mathrm{g}_T}. \tag{2.82}$$

The desired bound (2.75) will trivially follow once we establish

$$\sum_{\mathcal{S}\in\mathfrak{S}_n : \mathrm{Supp}\mathcal{S}\ni\tau} 1_{[(\mathcal{S},\sim)\text{ connected}]} \prod_{S\in\mathcal{S}} w_\epsilon(S) \leq C \tag{2.83}$$

because $(\mathcal{S}, \sim)$ is connected for any fusion $\mathcal{S}$. To prove (2.83), we first establish that, provided C_w is sufficiently large,

$$\sum_{S\sim S'} w_\epsilon(S) \mathrm{e}^{a(S)} \leq a(S'), \qquad a(S) := \begin{cases} |A| & \text{if } S \text{ is an } A\text{-set} \\ 1 & \text{if } S \text{ is a } J\text{-set} \end{cases} \tag{2.84}$$

Indeed, if S is an A-set, one uses (2.51) and if it is a J-set, one sums the exponential. Relying on (2.84) (which is a 'Kotecky-Preiss' criterion) in the terminology of Appendix A, we now apply the general combinatorial estimate Lemma A.1, obtaining

$$\sum_{\mathcal{S}\in\mathfrak{S}_n} 1_{[(\mathcal{S}\cup\{S'\},\sim)\text{ connected}]} \prod_{S\in\mathcal{S}} w_\epsilon(S) \leq \mathrm{e}^{a(S')} \tag{2.85}$$

(we bounded the indicator $1_{[\ldots]}$ by $k(\mathcal{S} \cup \{S'\})$ and we used (A4)). Consequently,

$$\begin{aligned} \sum_{\substack{\mathcal{S}\in\mathfrak{S}_n : \mathrm{Supp}\mathcal{S}\ni\tau \\ (\mathcal{S},\sim)\text{ connected}}} \prod_{S\in\mathcal{S}} w_\epsilon(S) \quad &\leq \quad \sum_{S' : S'\ni\tau} w_\epsilon(S') \sum_{\substack{\mathcal{S}\in\mathfrak{S}_n \\ (\mathcal{S}\cup\{S'\},\sim)\text{ connected}}} \prod_{S\in\mathcal{S}} w_\epsilon(S) \\ &\leq \quad \sum_{S' : S'\ni\tau} w_\epsilon(S') \mathrm{e}^{a(S')} \leq C \end{aligned} \tag{2.86}$$

This proves (2.83) and hence also (2.75).

The other claims of Lemma 2.4 are proven by analogous reasoning, using the bound on boundary polymers (2.52). Indeed, the general idea of the proof of (2.75) was to define w_ϵ in (2.80) such that it satisfies (2.83) and to extract a factor ϵ^{-1} from all terms in the sum on the LHS of (2.83), which was possible because all fusions $\mathcal{S}$

contain at least one A-set. To get (2.76), we modify the definition of w_ϵ for S such that $S \cap \{0, n+1\} \neq \emptyset$ to (note a new boundary-dependent constant $\breve{C}_w$)

$$w_\epsilon(S) := \begin{cases} d(A)^\alpha (\breve{C}_w)^{-1} (\breve{\epsilon} C_w)^{-|A \cap I_n|} \|G_A^c\|_\diamond & \text{if } S \text{ is an } A\text{-set} \\ C_w^{-1} \mathrm{e}^{-c_w |J|} & \text{if } S \text{ is a } J\text{-set} \end{cases} \tag{2.87}$$

and we use $\breve{\mathfrak{S}}_n, \breve{\mathfrak{S}}_n^f$ instead of $\mathfrak{S}_n, \mathfrak{S}_n^f$ by replacing in the definitions I_n by $\breve{I}_n$. Then (2.81) still holds with ϵ replaced by $\breve{\epsilon}$ provided that $\mathcal{S}$ contains at least one A-set with $|A \cap I_n| > 0$. This is the case unless $A' = \mathrm{Supp}\mathcal{S}$ is $\{0, n+1\}$ or $\breve{I}_n$, which are the two special cases in Lemma 2.4. Next, we need to establish (2.83) with C on the RHS replaced by $\breve{C}$ and $\tau = 0, n+1$. It is pedagogical to split this estimates according to which boundary times $\mathcal{S}$ contains. Let us consider the case where $0 \in \mathrm{Supp}\mathcal{S}$, but $n+1 \notin \mathrm{Supp}\mathcal{S}$, in particular choose $\tau = 0$. Then, we write, analogous to (2.86),

$$\sum_{\substack{\mathcal{S} \in \breve{\mathfrak{S}}_n : n+1 \notin \mathrm{Supp}\mathcal{S} \\ (\mathcal{S}, \sim) \text{ connected }, \, 0 \in \mathrm{Supp}\mathcal{S}}} \prod_{S \in \mathcal{S}} w_\epsilon(S) \quad \leq \quad \sum_{S' : S' \ni 0} w_\epsilon(S') \sum_{\substack{\mathcal{S} \in \mathfrak{S}_n \\ (\mathcal{S} \cup \{S'\}, \sim) \text{ connected}}} \prod_{S \in \mathcal{S}} w_\epsilon(S)$$

The sum over $\mathcal{S} \in \mathfrak{S}_n$ concerns only bulk quantities, and it is therefore bounded by $\mathrm{e}^{a(S')}$, just as before. We then conclude by bounding

$$\sum_{S' : S' \ni 0} w_\epsilon(S') \mathrm{e}^{a(S')} \leq \breve{C} \tag{2.88}$$

as follows from (2.52) when S is an A-set, otherwise it is the same bound as before. The cases where $\mathcal{S}$ contains $n+1$ but not 0, or $n+1$ and 0, are dealt with analogously, and we get (2.76). Finally, we turn to (2.77). The claim about $A = \{0, n+1\}$ follows directly from the bound on that polymer in Lemma 2.3. The claim concerning $A = \breve{I}_n$ differs from (2.76) because it is now possible that the fusion $\mathcal{S}$ consists of a single J-set $J = \breve{I}_n$, in which case one cannot extract $\breve{\epsilon}$. Taking this into account, this claim follows as above. □

3 From Hamiltonian dynamics to polymer models

In Section 2, the reasoning was largely independent of the details of the underlying Hamiltonian model. Indeed, in that section, we assumed some properties of the operator-valued polymers in Lemma 2.3 and we explored the consequences of these properties. Now, time has come to prove Lemma 2.3. This lemma discusses the ergodicity properties of the operator T and bounds on the correlation functions G_A^c. In both cases, the proof starts by expanding the microscopic evolution in a Dyson (or Duhamel) expansion. This expansion is standard and has been used many times in a related context. What might however seem odd at a first glance, is our complicated presentation of the perturbation series, involving tensor products of copies of the space $\mathscr{R} = \mathscr{B}(\mathscr{B}_1(\mathscr{H}_\mathrm{S}))$ (already introduced in Section 2.1.3). We use this formalism since we hope it makes the important estimates more natural and transparent. We first derive the expansion for G_A, G_A^c, see the expressions (3.17, 3.18). This part consists of purely algebraic manipulations (strictly speaking their validity is only established in Section 3.3.4 where we show that some series is absolutely summable).

The analysis part comes in Section 3.4, where we control $\|G_A^c\|_\diamond$. For reasons of readability, we first restrict ourselves to 'bulk' polymers. Afterwards, the analysis is repeated with minor adjustments to include the boundary polymers; this is done in Section 3.6-3.7.

As mentioned above, we also need to establish the exponential ergodicity of the operator T, which is done in Section 3.5. The crux of this argument is to relate T to the Markovian approximation, which was already discussed in the introductory Section 1.6.

3.1 Dyson series

Recall and rewrite the reduced dynamics Q_t, introduced in Section 2.1.2;

$$Q_t \rho_\mathrm{S} \quad = \quad \mathrm{Tr}_\mathrm{F} \left[\mathrm{e}^{-\mathrm{i}tL_\kappa} (\rho_\mathrm{S} \otimes P_\Omega) \right] = \mathbb{E} \left[\mathrm{e}^{-\mathrm{i}tL_\kappa} \right] \rho_\mathrm{S} = \mathbb{E} \left[\mathrm{e}^{\mathrm{i}tL_\mathrm{F}} \mathrm{e}^{-\mathrm{i}tL_\kappa} \right] \rho_\mathrm{S} \tag{3.1}$$

By the Duhamel formula (2.14) in Lemma 2.1, we get

$$\mathrm{e}^{\mathrm{i}tL_{\mathrm{S}}}Q_t\rho_{\mathrm{S}} \;=\; \sum_{m\in\mathbb{N}}(-1)^m \int\limits_{0<t_1<\ldots<t_{2m}<t} \mathrm{d}t_1\ldots\mathrm{d}t_{2m}\;\; \mathrm{Tr}_{\mathrm{F}}\left[L_{\mathrm{I},\kappa}(t_{2m})\ldots L_{\mathrm{I},\kappa}(t_2)L_{\mathrm{I},\kappa}(t_1)(\rho_{\mathrm{S}}\otimes P_\Omega)\right] \tag{3.2}$$

where, the RHS is understood to be 1 for $m=0$. Note that the RHS contains only terms with an even number of operators $L_{\mathrm{I},\kappa}(s)$ because $\mathrm{Tr}_{\mathrm{F}}(\Phi P_\Omega)=0$ whenever Φ is a monomial of odd degree in creation/annihilation operators. We use the Wick theorem to evaluate the expression in (3.2). To this purpose, we use the formalism developed in Section 2.1.3, as anticipated in Remark 2.2. For any $t_i, i=1,\ldots,2m$, we define a copy $\mathscr{R}_{t_i}$ of the space $\mathscr{R}$ (we do not aim to define "continuous" tensor products; all our formulas contain a finite number of t_i's). In particular, we write

$$\mathbb{E}\left[L_{\mathrm{I},\kappa}(t_{2m})\ldots L_{\mathrm{I},\kappa}(t_2)L_{\mathrm{I},\kappa}(t_1)\right] = \mathcal{T}\mathbb{E}\left[L_{\mathrm{I},\kappa}(t_{2m})\otimes_{\mathrm{S}}\ldots\otimes_{\mathrm{S}}L_{\mathrm{I},\kappa}(t_2)\otimes_{\mathrm{S}}L_{\mathrm{I},\kappa}(t_1)\right] \tag{3.3}$$

where, on the RHS, the operator $\mathcal{T}$ acts on operators in $\mathscr{R}_{\{t_1,\ldots,t_{2m}\}}=\otimes_{i=1}^{2m}\mathscr{R}_{t_i}$ and it contracts the operators such that those in $\mathscr{R}_{t_1}$ are on the right, then those on $\mathscr{R}_{t_2}$, etc.., as in (2.25).

Let $\{u,v\}$ be a pair of times with the convention that $u<v$. Then we define

$$K_{u,v} := -\mathbb{E}\left[L_{\mathrm{I},\kappa}(v)\otimes_{\mathrm{S}}L_{\mathrm{I},\kappa}(u)\right], \qquad K_{u,v}\in\mathscr{R}\otimes\mathscr{R} \tag{3.4}$$

which can be written more explicitly:

$$\begin{aligned} K_{u,v} \;&=\; -\lambda^2\mathrm{e}^{\kappa}h(v-u)\left[\mathcal{R}(D(v))\otimes\mathcal{L}(D(u))\right] - \lambda^2\mathrm{e}^{\kappa}h(u-v)\left[\mathcal{L}(D(v))\otimes\mathcal{R}(D(u))\right] \\ &+\; \lambda^2 h(v-u)\left[\mathcal{L}(D(v))\otimes\mathcal{L}(D(u))\right] + \lambda^2 h(u-v)\left[\mathcal{R}(D(v))\otimes\mathcal{R}(D(u))\right] \end{aligned} \tag{3.5}$$

where $h(-s)=\overline{h(s)}$, defined in Assumption A and $D(s)=\mathrm{e}^{\mathrm{i}sH_{\mathrm{S}}}D\mathrm{e}^{-\mathrm{i}sH_{\mathrm{S}}}$. For later use, we note that $\|K_{u,v}\|_\diamond \le \lambda^2 C|h(v-u)|$ for sufficiently small κ (for example, $C=4\|D\|\mathrm{e}^{\kappa}$). We view the operator $K_{u,v}$ as acting on the copies $\mathscr{R}_v\otimes\mathscr{R}_u$ (that is, we should in fact write $\boldsymbol{I}_{\{u,v\}}[K_{u,v}]$ but, as for the operators G_A^c, we prefer to drop the embedding operators $\boldsymbol{I}[\cdot]$.) Let us now apply the Wick theorem and expand any contribution to the integral in (3.2) in contractions of the creation and annihilation operators. This yields

$$\mathcal{T}\mathbb{E}\left[L_{\mathrm{I},\kappa}(t_{2m}))\otimes_{\mathrm{S}}\ldots\otimes_{\mathrm{S}}L_{\mathrm{I},\kappa}(t_1))\right] = \sum_{\pi\in\mathrm{Pair}(t_1,\ldots,t_{2m})}\mathcal{T}\left[\bigotimes_{\{u,v\}\in\pi}K_{u,v}\right] \tag{3.6}$$

where the sum on the RHS runs over pairings π, i.e. partitions of the times $t_1,\ldots,t_{2m}$ in m pairs $\{u,v\}$ with the notational convention that $u<v$. This formula relies crucially on the fact that the interaction is linear in creation/annihilation operators (nonetheless, it is easy to extend the proof so as to cover an additional small quadratic interaction). By plugging (3.6) into (3.2), we obtain

$$\mathrm{e}^{\mathrm{i}tL_{\mathrm{S}}}Q_t \;=\; \sum_{m\in\mathbb{N}}\int\limits_{0<t_1<\ldots<t_{2m}<t} dt_1\ldots\mathrm{d}t_{2m}\sum_{\pi\in\mathrm{Pair}(t_1,\ldots,t_{2m})}\mathcal{T}\left[\bigotimes_{\{u,v\}\in\pi}K_{u,v}\right] \tag{3.7}$$

where, for $m=0$, the RHS is understood as 1.

3.2 The evolution as an integral over time-pairs

We will now rewrite (3.7) in a convenient way. The integral over ordered $t_1,\ldots,t_{2m}$, together with the sum over pairings, π, on the set of times, is represented as an integral/sum over unordered pairs $\{u_i,v_i\}$ with $u_i,v_i\in\mathbb{R}_+$ and $i=1,\ldots,m$, such that

$$u_i<v_i, \qquad u_1<\ldots<u_m \tag{3.8}$$

This is done as follows. For any pair $\{t_r,t_s\}\in\pi$ with $r<s$, hence $t_r<t_s$, we let $u_i=t_r, v_i=t_s$ where the index $i=1,\ldots,m$ is chosen such that the u_i are ordered $u_1<u_2\ldots<u_m$. We write $w=\{u,v\}$ for a pair (with the convention that $u<v$) and $\underline{w}$ for a finite, possibly empty, collection of pairs. In the formulas below, we treat u,v as being implicitly defined by w. Given a Borel set $J\subset\mathbb{R}_+$, let Σ_J^m be the set of collections of m pairs

$\{u_i, v_i\}_{i=1,\dots,m}$ with $u_i, v_i \in J$ and satisfying the conditions (3.8). Let $\mu_m(d\underline{w})$ be the Lebesgue measure on the corresponding subset - a simplex - of J^{2m}. We can then rewrite

$$\int\limits_{0<t_1<\ldots<t_{2m}<t} dt_1 \ldots \mathrm{d}t_{2m} \sum_{\pi\in \mathrm{Pair}(t_1,\ldots,t_{2m})} \mathcal{T}\left[\bigotimes_{\{u,v\}\in\pi} K_{u,v}\right] = \int\limits_{\Sigma^m_{[0,t]}} \mu_m(d\underline{w})\mathcal{T}\left[\otimes_{w\in\underline{w}} K_w\right] \tag{3.9}$$

where we abbreviated $K_w := K_{u,v}$ for $w = \{u, v\}$. Let Σ_J denote the disjoint union

$$\Sigma_J = \sqcup_{m=0}^{\infty} \Sigma_J^m,$$

i.e. $\underline{w} \in \Sigma_J$ is given by $\underline{w} = (m, \underline{w}^m)$, $m \in \mathbb{N}$, $\underline{w}^m \in \Sigma_J^m$ with the convention $\Sigma_J^0 = \{\emptyset\}$ and we write $|\underline{w}| := m$. Thus measurable functions F on Σ_J are collections $\{F_m\}_{m\in\mathbb{N}}$ of measurable F_m on Σ_J^m. We let $\mu(\mathrm{d}\underline{w})$ be the measure on Σ_J given by

$$\int\limits_{\Sigma_J} \mu(\mathrm{d}\underline{w})F(\underline{w}) := \sum_{m\in\mathbb{N}} \int\limits_{\Sigma_J^m} \mu_m(d\underline{w})F_m(\underline{w}) \tag{3.10}$$

where we set $\mu_0((0,\emptyset)) = 1$. Note that the elements of Σ_J are naturally interpreted as sets, i.e. we write $w \in \underline{w}$ to mean $w \in \underline{w}^m$ for $m = |\underline{w}|$ and $\underline{w} = \underline{w}' \cup \underline{w}''$ for the element in Σ_J^m with $m = |\underline{w}'| \cup |\underline{w}''|$ with $w \in \underline{w}$ whenever $w \in \underline{w}'$ or $w \in \underline{w}''$. With these conventions the Dyson expansion (3.7) becomes

$$\mathrm{e}^{\mathrm{i}tL_{\mathrm{S}}}Q_t = \int\limits_{\Sigma_{[0,t]}} \mu(\mathrm{d}\underline{w})\mathcal{T}\left[\bigotimes_{w\in\underline{w}} K_w\right] \tag{3.11}$$

where, for $\underline{w} = \emptyset$ the integrand is defined to be 1.

3.3 Correlation functions and the Dyson series

3.3.1 The contraction operator $\mathcal{T}_A$

To each macroscopic time $\tau \in I_n$, we now associate a domain of microscopic times,

$$\mathrm{Dom}(\tau) = [\lambda^{-2}(\tau - 1), \lambda^{-2}\tau] \tag{3.12}$$

To a set $A \subset I_n$ of macroscopic times, we then associate the domain

$$\mathrm{Dom}(A) = \bigcup_{\tau\in A} \mathrm{Dom}(\tau) \tag{3.13}$$

The contraction operator $\mathcal{T}[\cdot]$ defined in Section 3.1 contracts operators so as to produce an operator in $\mathscr{R}$. We now define a contraction operator $\mathcal{T}_A$ that produces operators in $\mathscr{R}_A$. Let us first consider a finite family of operators $V_{t_i} \in \mathscr{R}_{t_i}$ where the indexed times t_i satisfy $t_i < t_{i+1}$ and $t_i \in \mathrm{Dom}(A)$. Then we set

$$\mathcal{T}_A\left[\bigotimes_i V_{t_i}\right] := \bigotimes_{\tau\in A} \boldsymbol{I}_\tau \left[\mathcal{T}\left[\bigotimes_{j:t_j\in\mathrm{Dom}(\tau)} V_{t_j}\right]\right] \tag{3.14}$$

and we extend by linearity to the whole of $\otimes_i \mathscr{R}_{t_i}$, obtaining $\mathcal{T}_A : \otimes_i \mathscr{R}_{t_i} \mapsto \mathscr{R}_A$. In words, $\mathcal{T}_A$ puts each operator into the right 'macroscopic' time-copy and contracts the operators within each macroscopic time-copy.

3.3.2 The graph $\mathcal{G}(\underline{w})$

A set of pairs $\underline{w} \in \Sigma_{\mathrm{Dom}(A)}$ determines a graph $\mathcal{G}(\underline{w})$ on I_n by the following prescription: the vertices $\tau < \tau'$ are connected by an edge iff. there is at least one pair $w = \{u, v\}$ in $\underline{w}$ such that

$$u \in \mathrm{Dom}(\tau) \qquad \text{and} \qquad v \in \mathrm{Dom}(\tau') \tag{3.15}$$

We write $\mathrm{Supp}(\mathcal{G}(\underline{w}))$ for the set of non-isolated vertices of $\mathcal{G}(\underline{w})$, i.e. the vertices that belong to at least one edge. If $\underline{w} \in \Sigma_{\mathrm{Dom}(A)}$ than $\mathrm{Supp}(\mathcal{G}(\underline{w}))$ is a subset of A. In that case, we write $\mathcal{G}_A(\underline{w})$ for the induced subgraph on A.

Let us denote the free S-evolutions

$$Y_\tau = \boldsymbol{I}_\tau[\mathrm{e}^{\mathrm{i}(\tau-1)L_{\mathrm{S}}}], \quad Y_A = \bigotimes_{\tau\in A} Y_\tau, \qquad \text{and} \qquad \widetilde{Y}_\tau = \boldsymbol{I}_\tau[\mathrm{e}^{-\mathrm{i}\tau L_{\mathrm{S}}}], \quad \widetilde{Y}_A = \bigotimes_{\tau\in A} \widetilde{Y}_\tau \tag{3.16}$$

We are now ready to state the connection between the Dyson expansion and the correlation functions G_A, G_A^c: For $A \subset I_n$ with $|A| \geq 2$,

$$\widetilde{Y}_A G_A Y_A \;=\; \int\limits_{\Sigma_{\mathrm{Dom}(A)}} \mu(\mathrm{d}\underline{w})\; 1_{[\mathrm{Supp}(\mathcal{G}(\underline{w}))=A]}\, \mathcal{T}_A\left[\bigotimes_{w\in\underline{w}} K_w\right] \tag{3.17}$$

$$\widetilde{Y}_A G_A^c Y_A \;=\; \int\limits_{\Sigma_{\mathrm{Dom}(A)}} \mu(\mathrm{d}\underline{w})\; 1_{[\mathcal{G}_A(\underline{w})\text{ connected}]}\, \mathcal{T}_A\left[\bigotimes_{w\in\underline{w}} K_w\right]. \tag{3.18}$$

To check this, it is helpful to note first (with $A = \{\tau_1, \ldots, \tau_m\}$)

$$\widetilde{Y}_A \mathbb{E}[U_{\tau_m} \otimes_{\mathrm{S}} \ldots \otimes_{\mathrm{S}} U_{\tau_1}] Y_A = \int\limits_{\Sigma_{\mathrm{Dom}(A)}} \mu(\mathrm{d}\underline{w})\; \mathcal{T}_A\left[\bigotimes_{w\in\underline{w}} K_w\right] \tag{3.19}$$

which is a straightforward generalization of (3.11). Then, one observes that all pairs $\{u, v\}$ with $u, v \in \mathrm{Dom}(\tau)$ make up $\boldsymbol{I}_\tau[T]$, and hence the operators G_A are integrals over all those $\underline{w}$ such that for each $\tau \in A$, the graph $\mathcal{G}(\underline{w})$ has an edge $\{\tau, \tau'\}$ containing τ. This leads to (3.17). Then, one verifies that G_A^c as given by (3.18) satisfies (2.27). Since (2.27) fixes G_A^c uniquely, (3.18) is thereby proven.

3.3.3 From pairings to G_A^c: Pictorial representation

We divide the time-axis into blocks (intervals of the form $\lambda^{-2}[\tau - 1, \tau]$ with $\tau \in \mathbb{N}$)). We draw pairings as arcs connecting two times on the axis, either solid  or dotted  , the reason to draw some of them differently is addressed below. The two times in a pairing, $\{u, v\}$, are denoted as 'legs'. This yields the upper picture in Figure (3).

Then, we single out blocks which **do not** contain a leg of a pair whose other leg lies in another block. Hence, such a block τ can contain no pairing at all, or one pairing, or two pairings, etc... In the language of the graph $\mathcal{G}(\underline{w})$, this means that the time τ has no edge to any other time; it is an isolated vertex. Such 'isolated blocks' are now, in the middle picture, drawn as  and we omit the pairs on them (imagining that they have been resummed and stands for the sum). Blocks which are not isolated are drawn as  , but we still indicate the pairs on them. It is now clear that the pairings that were drawn with a dotted line in the above picture, are those inside an isolated vertex and only the other ones (drawn with a solid line) are reproduced in the middle picture.

Finally, to arrive at the lower picture, we resum the pairings on the blocks  corresponding to connected components of the graph $\mathcal{G}(\underline{w})$. That, is whenever two blocks are connected by a pairing, we call them connected, and this induces a partition of the blocks into connected components. Only this partition is indicated in the lower figure. Blocks belonging to the same (connected component) determine one polymer A. Alternatively, the operator G_A^c is determined by summing all sets of pairs $\underline{w}$ that contribute to the connected component A.

3.3.4 Bounds on correlation functions

Up to now, we did nothing more on the Dyson expansions than straightforward algebraic manipulations. In what follows, we provide bounds. From the properties of the norm $\|\cdot\|_\diamond$ discussed in 2.1.7, we get

$$\|G_A^c\|_\diamond = \|\widetilde{Y}_A G_A^c Y_A\|_\diamond \leq \int\limits_{\Sigma_{\mathrm{Dom}(A)}} \mu(\mathrm{d}\underline{w}) \prod_{w\in\underline{w}} \|K_w\|_\diamond \tag{3.20}$$

To continue, we recall that $\|K_w\|_\diamond \leq \lambda^2 C|h(v-u)|$ (Section 3.1) and that $\int_0^\infty \mathrm{d}s(1+s)^\alpha |h(s)| < \infty$ by Assumption A. We say that '$\underline{w}$ spans A minimally' if the graph $\mathcal{G}_A(\underline{w})$ is connected and no pair w can be omitted from $\underline{w}$ without losing this property. In particular, if $\underline{w}$ spans A minimally, than $\mathcal{G}_A(\underline{w})$ is a spanning tree.

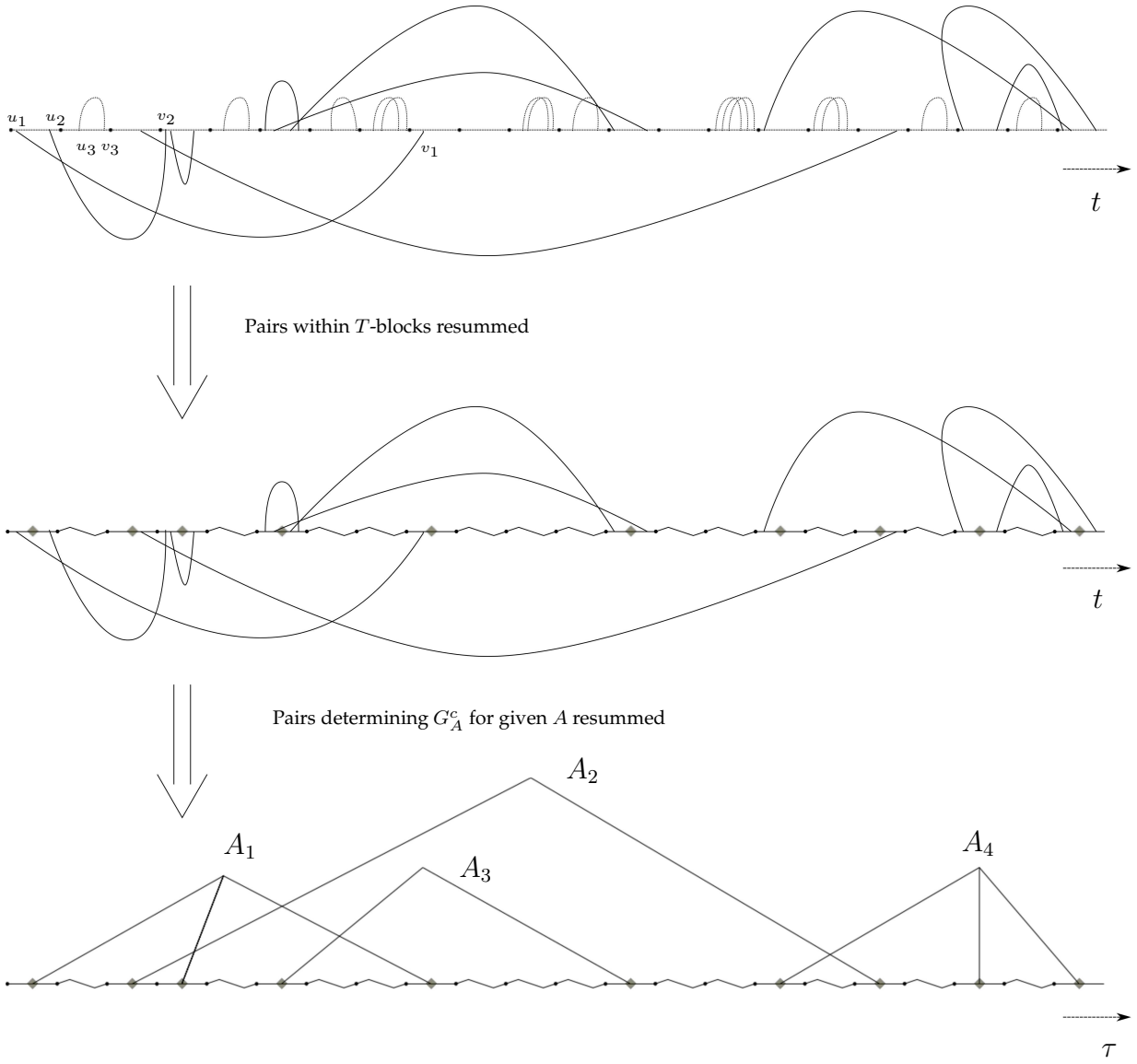

Figure 3: From the collection $\underline{w}$ of pairs $\{u_i, v_i\}$ to a collection $\mathcal{A}$ of polymers, $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$.

Lemma 3.1. *Let* $\|h\|_1 := \int_0^\infty \mathrm{d}s|h(s)|$. *For* $A \subset I_n$,

$$\|G_A^c\|_\diamond \leq \mathrm{e}^{C\|h\|_1|A|} \int_{\Sigma_{\mathrm{Dom}(A)}} \mu(\mathrm{d}\underline{w})\ 1_{[\underline{w}\ \textit{spans}\ A\ \textit{minimally}]} \left(\prod_{w \in \underline{w}} \lambda^2 C|h(v-u)| \right) \tag{3.21}$$

Proof. We start with an appealing estimate that was the main motivation for encoding the pairings π in the pair-sets $\underline{w}$. For integrable F,

$$\int\limits_{\Sigma_{\mathrm{Dom}(A)}} \mu(\mathrm{d}\underline{w})\, 1_{[\mathcal{G}_A(\underline{w})\ \text{connected}]}\, |F(\underline{w})| \quad \leq \quad \int\limits_{\Sigma_{\mathrm{Dom}(A)}} \mu(\mathrm{d}\underline{w}')\, 1_{[\underline{w}'\ \text{spans}\ A\ \text{minimally}]} \int\limits_{\Sigma_{\mathrm{Dom}(A)}} \mu(\mathrm{d}\underline{w}'')\, |F(\underline{w}' \cup \underline{w}'')| \tag{3.22}$$

To realize why this holds true, choose a spanning tree $\mathscr{T}$ for the connected graph $\mathcal{G}_A(\underline{w})$, pick a minimal subset

$\underline{w}'$ of the collection $\underline{w}$ such that $\mathcal{G}_A(\underline{w}') = \mathscr{T}$ and use that

$$\mu(\mathrm{d}(\underline{w}' \cup \underline{w}'')) = \mu(\mathrm{d}\underline{w}')\mu(d\underline{w}''). \tag{3.23}$$

We apply the inequality (3.22) to (3.20) with $F(\underline{w}) = \prod_{w\in\underline{w}} \|K_w\|_\diamond$ and we estimate the $\underline{w}''$-integral as follows:

$$\begin{aligned} & \int\limits_{\Sigma_{\mathrm{Dom}(A)}} \mu(\mathrm{d}\underline{w}'') \prod_{w\in\underline{w}''} \lambda^2 C|h(v-u)| \\ \leq & \sum_{m\in\mathbb{N}} \int\limits_{u_1<\ldots<u_m, u_i\in\mathrm{Dom}(A)} d\underline{u} \left(\prod_{i=1}^m \int_{u_i}^t \mathrm{d}v_i\, \lambda^2 C|h(v_i-u_i)| \right) \\ \leq & \sum_{m\in\mathbb{N}} \frac{(\lambda^2 C|\mathrm{Dom}(A)|\|h\|_1)^m}{m!} \leq \mathrm{e}^{\lambda^2 C|\mathrm{Dom}(A)|\|h\|_1} \end{aligned} \tag{3.24}$$

and we conclude, since $|\mathrm{Dom}(A)| = \lambda^{-2}|A|$. □

In what follows, we no longer trace explicitly the dependence on h and we simply estimate $\|h\|_1 \leq C$.

3.4 Proof of bound (2.51) in Lemma 2.3

In this section, we prove (2.51) of Lemma 2.3. We start from Lemma 3.1. For each $\underline{w}$ that spans A minimally, $\mathcal{G}_A(\underline{w})$ is a spanning tree on A. Hence we can reorganize the bound (3.21) by first integrating all $\underline{w}$ that determine the same spanning tree $\mathscr{T}$. This amounts to integrate, for each edge of the tree, all pairs $\{u, v\}$ that determine this edge. Hence we arrive at the bound

$$\|G_A^c\|_\diamond \leq \mathrm{e}^{C|A|} \sum_{\text{trees } \mathscr{T} \text{ spanning } A} \prod_{\{\tau,\tau'\}\in\mathcal{E}(\mathscr{T})} \hat{e}(\tau,\tau') \tag{3.25}$$

where $\mathcal{E}(\mathscr{T})$ is the set of edges of the tree $\mathscr{T}$ and, for $\tau' > \tau$,

$$\hat{e}(\tau,\tau') \;:=\; \int\limits_{\lambda^{-2}[\tau-1,\tau]} \mathrm{d}u \int\limits_{\lambda^{-2}[\tau'-1,\tau']} \mathrm{d}v\; \lambda^2 C|h(v-u)|, \tag{3.26}$$

and $\hat{e}(\tau',\tau) := \hat{e}(\tau,\tau')$. Let $\Delta\tau \equiv \tau' - \tau$ and $s \equiv v - u$, then

$$\hat{e}(\tau,\tau') \;\leq\; C \int\limits_{(\Delta\tau-1)/\lambda^2}^{(\Delta\tau+1)/\lambda^2} \mathrm{d}s|h(s)|, \qquad \Delta\tau > 1 \tag{3.27}$$

$$\hat{e}(\tau,\tau') \;\leq\; \lambda^2 C \int_0^{2/\lambda^2} \mathrm{d}s\, s|h(s)|, \qquad \Delta\tau = 1 \tag{3.28}$$

and, since $\int_0^\infty \mathrm{d}s(1+|s|)^\alpha|h(s)| \leq C$, this implies the bound

$$\sum_{\tau':\tau'\neq\tau} \hat{e}_\alpha(\tau,\tau') \leq C\epsilon \tag{3.29}$$

where $\epsilon = |\lambda|^{2\min(\alpha,1)}$ (cfr. Lemma 3.1) and

$$\hat{e}_\alpha(\tau,\tau') := (1+|\tau'-\tau|)^\alpha \hat{e}(\tau,\tau'). \tag{3.30}$$

Starting from (3.25), we bound

$$\begin{aligned} \sum_{A:A\ni\tau_0} (C\epsilon)^{-(|A|-1)} d(A)^\alpha \|G_A^c\|_\diamond \;&\leq\; \sum_{A:A\ni\tau_0} (C\epsilon)^{-(|A|-1)} \sum_{\text{trees } \mathscr{T} \text{ spanning } A} \prod_{\{\tau,\tau'\}\in\mathcal{E}(\mathscr{T})} \hat{e}_\alpha(\tau,\tau') \\ &\leq \sum_{\text{trees } \mathscr{T}:\tau_0\in\mathcal{V}(\mathscr{T})} (C\epsilon)^{-|\mathcal{E}(\mathscr{T})|} \prod_{\{\tau,\tau'\}\in\mathcal{E}(\mathscr{T})} \hat{e}_\alpha(\tau,\tau') \end{aligned} \tag{3.31}$$

where $\mathcal{E}(\mathscr{T}), \mathcal{V}(\mathscr{T})$ are the edge, resp. vertex set of the tree $\mathscr{T}$. To obtain the first inequality, we used that $\prod_{\{\tau,\tau'\}\in\mathcal{E}(\mathscr{T})}(1+|\tau'-\tau|)^\alpha > d(A)^\alpha$ for any spanning tree on A. The sum over trees in (3.31) is estimated with the help of Lemma A.1 by choosing

1) the polymers S as unordered pairs $\{\tau,\tau'\}$ and $a(S)=1$.
2) the adjacency relation $S\sim S'$ iff. $S\cap S'\neq\emptyset$.
3) the polymer weights $w(\{\tau,\tau'\}) = (C\epsilon)^{-1}\hat{e}_\alpha(\tau,\tau')$

The bound (3.29) plays the role of the Kotecky-Preiss criterion (A7) and Lemma A.1 yields

$$\sup_{\tau_0}\sum_{\text{trees }\mathscr{T}:\tau_0\in\mathcal{V}(\mathscr{T})}(C\epsilon)^{-|\mathcal{E}(\mathscr{T})|}\prod_{\{\tau,\tau'\}\in\mathcal{E}(\mathscr{T})}\hat{e}_\alpha(\tau,\tau')\leq 1 \tag{3.32}$$

For example, replace the restriction $\tau_0\in\mathcal{V}(\mathscr{T})$ by the weaker restriction $\mathscr{E}(\mathscr{T})\sim\{\tau_0,\tau_1\}$ for some arbitrary τ_1 and use (A3). Upon combining this with (3.31), the bound (2.51) of Lemma 2.3 is proven.

3.5 Analysis of T: dissipativity and weak coupling limit

Recall that operator T, acting on $\mathscr{B}_1(\mathscr{H}_\mathrm{S})$, that was introduced in Section 2.1.2

$$T\rho_\mathrm{S} = \mathrm{Tr}_\mathrm{F}[\mathrm{e}^{-\mathrm{i}\lambda^{-2}L_\kappa}(\rho_\mathrm{S}\otimes P_\Omega)] \tag{3.33}$$

We exhibit the dissipative properties of T and we establish Statement 1) of Lemma 2.3.

3.5.1 Construction of Lindblad generator M

To ease the presentation, we introduce some shorthand notation. Recall the one-dimensional spectral projections P_e corresponding to H_S (see the discussion preceding Assumption B). The set of differences of eigenvalues will be called $\mathcal{E}=\{\varepsilon=e-e', e,e'\in\sigma(H_\mathrm{S})\}$. Note that it is the set of eigenvalues of the Liouvillian L_S. By our non-degeneracy assumption, we associate a unique pair $(e,e'), e,e'\in\sigma(H_\mathrm{S})$ to any $\varepsilon\in\mathcal{E}, \varepsilon\neq 0$ by $\varepsilon=e-e'$. We then abbreviate $D_\varepsilon\equiv P_e D P_{e'}$ and we note that $D_\varepsilon^* = D_{-\varepsilon}$ since $D=D^*$. We define the operator M, acting on $\mathscr{B}_1(\mathscr{H}_\mathrm{S})$

$$M\rho = -\mathrm{i}[H_\mathrm{Lamb},\rho] + \sum_{\varepsilon\in\mathcal{E}:\varepsilon<0}\hat{h}(-\varepsilon)\left(\mathrm{e}^\kappa D_\varepsilon\rho D_\varepsilon^* - \frac{1}{2}\{D_\varepsilon^* D_\varepsilon,\rho\}\right) \tag{3.34}$$

with $\hat{h}(\varepsilon)$ as in (1.9), $\{A,B\}=AB+BA$, and the energy-shift ('Lamb-shift') operator

$$H_\mathrm{Lamb} = \sum_{\varepsilon\in\mathcal{E}:\varepsilon\neq 0}\left(\mathrm{Im}\int_0^\infty \mathrm{d}s\,\mathrm{e}^{\mathrm{i}s\varepsilon}h(s)\right)D_\varepsilon^* D_\varepsilon, \tag{3.35}$$

which is well-defined by Assumption A).

Because of our strong non-degeneracy assumptions, we can greatly simplify the form of M. Let us choose a basis for $\mathscr{B}(\mathscr{H}_\mathrm{S})$ consisting of the one-dimensional projectors P_e and the operators $|\psi_e\rangle\langle\psi_{e'}|$, with $e\neq e'$, where $\psi_{e\in\sigma(H_\mathrm{S})}$ are eigenfunctions of H_S. Then, inspection of (3.34) and the definition of the jump rates $j(\cdot,\cdot)$ in (1.9) yields the following claims:

Off-diagonal elements The operators $|\psi_e\rangle\langle\psi_{e'}|$ are eigenvectors of M with corresponding eigenvalue

$$-\mathrm{i}\langle\psi_e, H_\mathrm{Lamb}\psi_e\rangle - \mathrm{i}\langle\psi_{e'}, H_\mathrm{Lamb}\psi_{e'}\rangle - (1/2)\sum_{e''}(j(e',e'')+j(e,e'')) \tag{3.36}$$

Assumption B implies that the second term is bounded away from 0 and hence one has

$$\sup_{e\neq e'}\|\mathrm{e}^{tM}(|\psi_e\rangle\langle\psi_{e'}|)\|\leq\mathrm{e}^{-ct}$$

with $c>0$. This corresponds physically to decoherence.

Diagonal elements The space spanned by the projectors P_e is mapped into itself and we can identify the action of

M on this space with that of a $d_{\rm S}\times d_{\rm S}$-matrix $\mathcal{M}$ (with $d_{\rm S}=\dim\mathscr{H}_{\rm S}$) by setting $M(\sum_e \mu(e)P_e)=\sum_e((\mathcal{M}\mu)(e))P_e$ for $\mu\in\mathbb{C}^{\rm d_S}$. Then

$$(\mathcal{M}\mu)(e)=\sum_{e'}(\mathrm{e}^{\kappa}j(e',e)\mu(e')-j(e,e')\mu(e)) \tag{3.37}$$

It is clear that, for $\kappa=0$, $\mathcal{M}$ is the forward generator of a Markov process with state space $\sigma(H_{\rm S})$. We determine the spectrum of $\mathcal{M}$. Recall from the discussion following Assumption A that $j(e',e)=0$ whenever $e\geq e'$, hence $\mathcal{M}$ is a triangle matrix and its singular values are the diagonal elements. By inspection of (3.37) and Assumption B, it follows that all singular values lie in the region $\{z:\mathrm{Re}\,z<-g_{\mathcal{M}}\}$ with $g_{\mathcal{M}}=\min_{e:e\neq e_0}\sum_{e'}j(e,e')$ save for a simple eigenvalue 0, corresponding to the eigenvector $\mu(e)=\delta_{e,e_0}$. By the spectral mapping theorem for semigroups with bounded generator, we then get that $\sigma(\mathrm{e}^{t\mathcal{M}})\subset\{0\}\cup\{z:|z|\leq\mathrm{e}^{-tg_{\mathcal{M}}}\}$

Note that the right eigenvector does not depend on κ, but the left eigenvector does. The Markov process at $\kappa=0$ is called absorbing, with absorbing state e_0. Physically, this property is due to the fact that the field F is in the vacuum state and cannot excite the atom S. We extract from this discussion what we need later on:

Lemma 3.2. *The ground-state projection P_{e_0} is an eigenvector of e^{tM} with simple eigenvalue 1. The rest of the spectrum of e^{tM} lies in the region $\{z:|z|\leq\mathrm{e}^{-tg_M}\}$ for some $g_M>0$ which does not depend on κ.*

3.5.2 The Lindblad generator M and the microscopic model

The relation between the the semigroup generated by the Lindblad generator M and the microscopic model is a classical result in mathematical physics. It was discussed already in Section 1.6.

Proposition 3.3. *For any $t>0$,*

$$\|Q_t-\mathrm{e}^{-\mathrm{i}tL_{\rm S}+\lambda^2tM}\|\leq C\mathrm{e}^{C\lambda^2t}|\lambda|^{2\alpha_*} \tag{3.38}$$

where $\alpha_*=\min(1,\alpha)$ for $\alpha\neq1$, and $|\lambda|^{2\alpha_*}=\lambda^2|\log|\lambda||$ for $\alpha=1$. Using that $L_{\rm S}$ commutes with M, we get

$$\lim_{\lambda\to0}\mathrm{e}^{\mathrm{i}\lambda^{-2}\mathfrak{t}L_{\rm S}}Q_{\lambda^{-2}\mathfrak{t}}=\mathrm{e}^{\mathfrak{t}M} \tag{3.39}$$

for any $\mathfrak{t}>0$.

For completeness, we review the simple proof of this convergence in Appendix B. We now turn to the

Proof of Statement 1) of Lemma 2.3. Note

i.) $L_{\rm S}$ commutes with and M.

ii.) $\mathrm{e}^{\mathrm{i}tL_{\rm S}}$ is an isometry on $\mathscr{B}_1(\mathscr{H}_{\rm S})$.

iii.) $L_{\rm S}(P_e)=0$ for any $e\in\sigma(H_{\rm S})$, in particular e_0.

Therefore, the spectral analysis of M (Lemma 3.2) implies that $\mathrm{e}^{-\mathrm{i}\lambda^{-2}L_{\rm S}+M}$ has a simple eigenvalue 1 and all other spectrum lies inside a circle of radius e^{-g_M}. On the other hand, recall that $T=Q_{\lambda^{-2}}$ and hence by Proposition 3.3, $T-\mathrm{e}^{-\mathrm{i}\lambda^{-2}L_{\rm S}+M}$ vanishes as $\lambda\to0$. We apply spectral perturbation theory of isolated eigenvalues to conclude that, for sufficiently small $|\lambda|,|\kappa|$, T has an isolated eigenvalue e^{θ_T} with corresponding spectral projector R such that statement 1) of Lemma 2.3 holds. Since T conserves the trace for $\kappa=0$, it follows that $\theta_T(\lambda,\kappa)$ indeed vanishes for $\kappa=0$; as $\lambda\to0$, it reduces to the eigenvalue of M, hence to 0, as well. Since T preserves positivity and the trace for $\kappa=0$, we can choose $\tilde\eta=\mathbb{1}$ and η a density matrix, where $\eta,\tilde\eta$ were defined as the right, resp. left eigenvectors of T (see beginning of Section 2.3). For $\kappa\in\mathbb{R}$, the operator T remains positivity-preserving and therefore $\eta,\tilde\eta$ can be chosen to be positive operators. □

For later use, we note that this perturbation argument also yields the bounds, for sufficiently small $|\lambda|,|\kappa|$,

$$\|\tilde\eta-\mathbb{1}\|\leq\mathcal{O}(|\kappa|),\qquad\|\eta-P_{e_0}\|\leq C|\lambda|^{2\alpha_*}. \tag{3.40}$$

3.6 Boundary polymers

3.6.1 Dyson expansion

We aim to write the analogue of the expansion (3.11) for the expression of $\breve{Q}_n\rho_{\mathrm{S}}$ given in (2.36). Recalling the definition (2.10), we may write $U_{\ltimes}$ in the same form as (2.14):

$$\begin{aligned} U_{\ltimes}(\rho_{\mathrm{S}}\otimes P_\Omega) &= \rho_{\mathrm{S}}\otimes \mathrm{e}^{\mathrm{i}\Phi_\kappa(\psi_{\ltimes},0)}P_\Omega \mathrm{e}^{-\mathrm{i}\Phi_{-\kappa}(\psi_{\ltimes},0)} \\ &= \sum_{m\in\mathbb{N}}(-\mathrm{i})^m \int\limits_{-1\le t_1<\ldots<t_m\le 0} \mathrm{d}t_1\ldots\mathrm{d}t_m\, L_{\mathrm{I},\kappa}(t_m)\ldots L_{\mathrm{I},\kappa}(t_2)L_{\mathrm{I},\kappa}(t_1)\rho_{\mathrm{S}}\otimes P_\Omega \end{aligned} \tag{3.41}$$

with the definition

$$L_{\mathrm{I},\kappa}(s) = -\mathcal{L}(\mathbb{1}\otimes\Phi_\kappa(\psi_{\ltimes},0)) + \mathcal{R}(\mathbb{1}\otimes\Phi_{-\kappa}(\psi_{\ltimes},0)), \qquad s\in[-1,0). \tag{3.42}$$

In the same way we get

$$U_{\rtimes} = \sum_{m\in\mathbb{N}}(-\mathrm{i})^m \int\limits_{n/\lambda^2\le t_1<\ldots<t_m\le n/\lambda^2+1} \mathrm{d}t_1\ldots\mathrm{d}t_m\, L_{\mathrm{I},\kappa}(t_m)\ldots L_{\mathrm{I},\kappa}(t_2)L_{\mathrm{I},\kappa}(t_1) \tag{3.43}$$

where

$$L_{\mathrm{I},\kappa}(s) = -\mathcal{L}(\mathbb{1}\otimes\Phi(\mathrm{e}^{\mathrm{i}n|q|/\lambda^2}\psi_{\rtimes},0)), \qquad s\in(n/\lambda^2, n/\lambda^2+1)]. \tag{3.44}$$

Thus we end up with a similar series as in (3.2) and in (3.7):

$$\begin{aligned} \mathrm{e}^{\mathrm{i}n/\lambda^2 L_{\mathrm{S}}}\breve{Q}_n\rho_{\mathrm{S}} &= \sum_{m\in\mathbb{N}}(-1)^m \int\limits_{-1\le t_1<\ldots<t_{2m}<n/\lambda^2+1} \mathrm{d}t_1\ldots\mathrm{d}t_{2m}\ \mathrm{Tr}_{\mathrm{F}}\left[L_{\mathrm{I},\kappa}(t_{2m})\ldots L_{\mathrm{I},\kappa}(t_2)L_{\mathrm{I},\kappa}(t_1)(\rho_{\mathrm{S}}\otimes P_\Omega)\right] \\ &= \sum_{m\in\mathbb{N}} \int\limits_{-1\le t_1<\ldots<t_{2m}\le n/\lambda^2+1} dt_1\ldots\mathrm{d}t_{2m} \sum_{\pi\in\mathrm{Pair}(t_1,\ldots,t_{2m})} \mathcal{T}\left[\bigotimes_{\{u,v\}\in\pi} K_{u,v}\right] \\ &= \int\limits_{\Sigma_{[-1,n/\lambda^2+1]}} \mu(\mathrm{d}\underline{w})\mathcal{T}\left[\bigotimes_{w\in\underline{w}} K_w\right] \end{aligned} \tag{3.45}$$

where $K_{u,v}$ is defined by the formula (3.4), now with u,v in $[-1,n/\lambda^2+1]$ instead of $[0,n/\lambda^2]$. Note that the choice of time-intervals $[-1,0]$ and $[n/\lambda^2,n/\lambda^2+1]$ is somewhat arbitrary; we do this to display the boundary terms in a way that resembles the bulk terms as close as possible.

3.6.2 Correlation functions

First, we extend the definition of the graph $\mathcal{G}(\underline{w})$ by allowing $A\subset\breve{I}_n$ and setting

$$\mathrm{Dom}(0) = [-1,0], \qquad \mathrm{Dom}(n+1) = [n/\lambda^2, n/\lambda^2+1] \tag{3.46}$$

We generalize (3.16) to $\tau = 0, n+1$ by putting

$$Y_0 = \widetilde{Y}_0 = \boldsymbol{I}_0[\mathbb{1}], \qquad Y_{n+1} = \boldsymbol{I}_{n+1}[\mathrm{e}^{\mathrm{i}(n/\lambda^2)L_{\mathrm{S}}}], \qquad \widetilde{Y}_{n+1} = \boldsymbol{I}_{n+1}[\mathrm{e}^{-\mathrm{i}(n/\lambda^2)L_{\mathrm{S}}}] \tag{3.47}$$

and then defining $Y_A, \widetilde{Y}_A$ with $A\subset\breve{I}_n$ as in (3.16). With these definitions, the formulae (3.17), (3.18) hold generally for $A\subset\breve{I}_n$;

$$\widetilde{Y}_A G_A Y_A = \int\limits_{\Sigma_{\mathrm{Dom}(A)}} \mu(\mathrm{d}\underline{w})1_{[\mathrm{Supp}\mathcal{G}_A(\underline{w})=A]}\mathcal{T}_A\left[\bigotimes_{w\in\underline{w}} K_w\right] \tag{3.48}$$

$$\widetilde{Y}_A G_A^c Y_A = \int\limits_{\Sigma_{\mathrm{Dom}(A)}} \mu(\mathrm{d}\underline{w})1_{[\mathcal{G}_A(\underline{w})\text{ connected}]}\mathcal{T}_A\left[\bigotimes_{w\in\underline{w}} K_w\right] \tag{3.49}$$

3.7 Bounds on boundary polymers

In this section, we prove the bound (2.52) in Lemma 2.3. We first generalize Lemma 3.1 to read

Lemma 3.4. *For $A \subset \breve{I}_n$*

$$\|G_A^c\|_\diamond \;\leq\; \breve{C}\mathrm{e}^{C|A|} \int\limits_{\Sigma_{\mathrm{Dom}(A)}} \mu(\mathrm{d}\underline{w}) \left(\prod_{w\in\underline{w}} \|K_w\|_\diamond \right) 1_{[\underline{w}\,\textit{spans A minimally}]} \tag{3.50}$$

Proof. To obtain bounds on the boundary pairings, we first note that (recall $\mathrm{Dom}(I_n) = [0, n/\lambda^2]$)

$$\|K_{u,v}\|_\diamond \leq \begin{cases} C|\lambda h_{\ltimes}(v)| & u \in \mathrm{Dom}(0), v \in \mathrm{Dom}(I_n) \\ C|\lambda h_{\rtimes}(n/\lambda^2 - u)| & u \in \mathrm{Dom}(I_n), v \in \mathrm{Dom}(n+1) \\ C|h_{\bowtie}(n/\lambda^2)| & u \in \mathrm{Dom}(0), v \in \mathrm{Dom}(I_n) \\ C\|\psi_{\ltimes}\|^2 & u, v \in \mathrm{Dom}(0) \\ C\|\psi_{\rtimes}\|^2 & u, v \in \mathrm{Dom}(n+1) \end{cases} \tag{3.51}$$

We proceed as in the proof of Lemma 3.1 and extract from each set of pairs $\underline{w}$ a minimally spanning subset $\underline{w}'$. To bound the $\underline{w}'' = \underline{w} \setminus \underline{w}'$-integral in (3.22), we need an estimate on

$$\int_{\Sigma_{\mathrm{Dom}(A)}} \mu(\mathrm{d}\underline{w}) \prod_{w\in\underline{w}} \|K_w\|_\diamond. \tag{3.52}$$

Denote, for $w = \{u, v\}$,

$$\begin{aligned} \chi_{\ltimes}(w) &:= 1_{u\in\mathrm{Dom}(0)} 1_{v\notin\mathrm{Dom}(n+1)} \\ \chi_{\rtimes}(w) &:= 1_{u\notin\mathrm{Dom}(0)} 1_{v\in\mathrm{Dom}(n+1)} \\ \chi_{\bowtie}(w) &:= 1_{u\in\mathrm{Dom}(0)} 1_{v\in\mathrm{Dom}(n+1)} \end{aligned} \tag{3.53}$$

and $\chi_\iota(\underline{w}) := \prod_{w\in\underline{w}} \chi_\iota(w)$ for $\iota = \ltimes, \rtimes, \bowtie$. For any $\underline{w} \in \mathrm{Dom}(A)$, we can write $\underline{w} = \underline{w}' \cup \underline{w}_{\ltimes} \cup \underline{w}_{\rtimes} \cup \underline{w}_{\bowtie}$ such that $\underline{w}' \in \mathrm{Dom}(A \cap I_n)$ and $\chi_\iota(\underline{w}_\iota) = 1$. Hence (3.23) implies

$$(3.52) \leq \left(\int\limits_{\Sigma_{\mathrm{Dom}(A\cap I_n)}} \mu(\mathrm{d}\underline{w}) \prod_{w\in\underline{w}} \|K_w\|_\diamond \right) \left(\prod_{\iota\in\{\ltimes,\rtimes,\bowtie\}} \int\limits_{\Sigma_{\mathrm{Dom}(A)}} \mu(\mathrm{d}\underline{w}) \chi_\iota(\underline{w}) \prod_{w\in\underline{w}} \|K_w\|_\diamond \right). \tag{3.54}$$

The first integral is estimated in (3.24), and the three others in the same way using the bounds (3.51). Thus, the $\iota = \ltimes, \rtimes$ -factors are bounded by

$$\sum_{m\in\mathbb{N}} \frac{1}{m!} \left(C\|\psi_\iota\|^2 + C \int_{[0,n/\lambda^2]} \mathrm{d}s\, |\lambda h_\iota(s)| \right)^m \leq \mathrm{e}^{|\lambda| C\|h_\iota\|_1 + C\|\psi_\iota\|^2}$$

and the $\iota = \bowtie$ by $\exp(C|h_{\bowtie}(n/\lambda^2)|)$. Altogether we conclude that the product (3.54), and hence (3.52), is bounded by $\breve{C}\mathrm{e}^{C|A|}$. □

To establish the bound (2.52), we closely follow the proof of (2.51) given in Section 3.4. First, we apply the definition of the edge factors $\hat{e}(\tau, \tau') = \hat{e}(\tau', \tau)$ in (3.26) to the case where $\{\tau, \tau'\} \not\subset I_n$; explicitly,

$$\hat{e}(\tau, \tau') \;:=\; \begin{cases} C \int\limits_{\mathrm{Dom}(\tau')} \mathrm{d}s\, |\lambda h_{\ltimes}(s)| & \tau = 0, \tau' \in I_n \\ C \int\limits_{\mathrm{Dom}(\tau')} \mathrm{d}s\, |\lambda h_{\rtimes}(n/\lambda^2 - s)| & \tau = n+1, \tau' \in I_n \\ C\ |h_{\bowtie}(n/\lambda^2)| & \tau = 0, \tau' = n+1 \end{cases} \tag{3.55}$$

and then also $\hat{e}_\alpha(\tau,\tau') := (1+|\tau'-\tau|)^\alpha \hat{e}(\tau,\tau')$. These boundary edge factors satisfy

$$\hat{e}_\alpha(0,n+1) \le \breve{C}, \qquad \text{and} \qquad \sum_{\tau'\in I_n} \hat{e}_\alpha(\tau,\tau') \le |\lambda|\breve{C}, \qquad \text{for } \tau = 0, n+1 \tag{3.56}$$

as follows immediately from the properties of $h_{\ltimes}, h_{\rtimes}, h_{\bowtie}$ in Assumption C. To get (2.52), we first restrict ourselves to A with $0 \in A$ but $n+1 \notin A$ (in particular, $\tau = 0$ in (2.52)). Starting from (3.50) and employing the definitions of $\hat{e}_\alpha(\tau,\tau')$ above, we proceed as in (3.31) to derive

$$\frac{1}{\breve{C}} \sum_{A\in\breve{I}_n : A\ni 0, n+1\notin A} (C\breve{\epsilon})^{-|A\cap I_n|} d(A)^\alpha \|G_A^c\|_\diamond \;\le \sum_{\text{trees } \mathscr{T}: 0\in\mathcal{V}(\mathscr{T}), n+1\notin\mathcal{V}(\mathscr{T})} (C\breve{\epsilon})^{-|\mathcal{E}(\mathscr{T})|} \prod_{(\tau,\tau')\in\mathcal{E}(\mathscr{T})} \hat{e}_\alpha(\tau,\tau')$$

where we used $|A\cap I_n| \ge |\mathscr{E}(\mathscr{T})|$. For every tree $\mathscr{T}$ in this sum, we identify vertices $\tau_1 < \tau_2 < \ldots < \tau_m$ such that $\{0,\tau_j\} \in \mathscr{E}(\mathscr{T})$ and $m \ge 1$. Then the sum over $\mathscr{T}$ is recast as a sum over m, choice of edges $\tau_1 < \tau_2 < \ldots < \tau_m$, and trees growing out of them (including the possibility of no tree). This yields the bound

$$\sum_{m\ge 1} \sum_{\tau_1<\ldots<\tau_m} \prod_{j=1}^m (C\breve{\epsilon})^{-1} \hat{e}_\alpha(0,\tau_j) \left(1 + \sum_{\text{trees } \mathscr{T}: \tau_j\in\mathcal{V}(\mathscr{T})\subset I_n} (C\breve{\epsilon})^{-|\mathcal{E}(\mathscr{T})|} \prod_{(\tau,\tau')\in\mathcal{E}(\mathscr{T})} \hat{e}_\alpha(\tau,\tau')\right) \tag{3.57}$$

$$\le \sum_{m\ge 1} \sum_{\tau_1<\ldots<\tau_m} \prod_{j=1}^m 2(C\breve{\epsilon})^{-1} \hat{e}_\alpha(0,\tau_j) \le \sum_{m\ge 1} \frac{1}{m!} \left(\sum_{\tau\in I_n} 2(C\breve{\epsilon})^{-1} \hat{e}_\alpha(0,\tau)\right)^m \le \breve{C} \tag{3.58}$$

The first inequality follows by (3.32), since the trees are in the bulk, the last inequality uses (3.56). The bound over A that do contain $n+1$ but not 0 and A that contain both $n+1$ and 0, is analogous. Note however that for A that contain both $n+1$ and 0 we cannot extract an $\breve{\epsilon}$ factor for every edge since some trees contain the edge $\{0,n+1\}$ (see in (3.56)). Therefore, the power of $(\breve{\epsilon})^{-1}$ in (2.52) is not $|A|-1$ but $|A\cap I_n|$.

4 Analysis of polymer model: Proof of main results

4.1 Proof of Theorem 1.1: Approach to steady state

To prove Theorem 1.1, we first exhibit approach to a steady state for discrete times of the form $t = n/\lambda^2$ and for a restricted class of initial states and observables (Section 4.1.1). Then we eliminate these restrictions in Sections 4.1.2, 4.1.3 and 4.1.4. The reasoning in Section 4.1.1 is based heavily on cluster expansions and we advise the reader to read Appendix A before continuing.

4.1.1 Approach to steady state for discrete times

We start from the polymer representation (2.71) with $\kappa = 0$, hence $\theta_T = 0$,

$$Z_n = k_{\ltimes} k_{\rtimes} \sum_{\mathcal{A}\in\breve{\mathfrak{B}}_n^1} \prod_{A\in\mathcal{A}} v(A) \tag{4.1}$$

where $Z_n = Z_n(O,\rho_0,\kappa=0)$ depends on the initial state and observable via $k_{\ltimes}, k_{\rtimes}$ and the weights $v(A)$ for $A\cap\{0,n+1\} \neq \emptyset$.

We write $A \sim A'$ whenever $\mathrm{dist}(A,A') \le 1$, and for a collection $\mathcal{A}$, $\mathcal{A} \sim A'$ whenever there is at least one $A\in\mathcal{A}$ such that $A \sim A'$ (i.e., $\mathrm{Supp}\mathcal{A} \sim A'$). These definitions coincide with those in Section A.2 of Appendix A.

We separate each collection $\mathcal{A}$ into its boundary and bulk polymers by writing

$$Z_n = k_{\ltimes} k_{\rtimes} \sum_{\substack{\breve{\mathcal{A}}\in\breve{\mathfrak{B}}_n^1 \\ \forall A\in\breve{\mathcal{A}}: A\cap\{0,n+1\}\neq\emptyset}} \left(\prod_{A\in\breve{\mathcal{A}}} v(A)\right) Z_{n,\mathrm{Supp}\breve{\mathcal{A}}} \tag{4.2}$$

where

$$Z_{n,A'} := \sum_{\substack{\mathcal{A} \in \mathfrak{B}_n^1 \\ \mathcal{A} \nsim A'}} \prod_{A \in \mathcal{A}} v(A) \tag{4.3}$$

Note that $Z_{n,A'}$ depends only on bulk polymer weights and that $\breve{\mathcal{A}}$ consist maximally of two sets, one containing the element 0 and one containing $n+1$. By the identity (2.64) with $\kappa = 0$,

$$Z_{n,\emptyset} = Z_n(\mathbb{1}, \eta \otimes P_\Omega, \kappa = 0) \tag{4.4}$$

By unitarity and the fact that $\mathrm{Tr}_{\mathrm{S}}\, \eta = 1$, we have

$$Z_{n,\emptyset} = \mathrm{Tr}[\mathrm{e}^{-\mathrm{i}(n/\lambda^2)H}(\eta \otimes P_\Omega)\mathrm{e}^{\mathrm{i}(n/\lambda^2)H}] = \mathrm{Tr}(\eta \otimes P_\Omega) = 1 \tag{4.5}$$

The quantity $Z_{n,A'}$ can be viewed as the partition function Υ_n of a polymer gas with polymer weights $w(A) \equiv v(A)1_{[A \nsim A']}$ (i.e. it is of the form (A5) In Section A.2). For ϵ small enough, the Kotecky-Preiss criterion (A7) is satisfied with $\delta \equiv C\epsilon$ because of (2.75) with $\alpha = 0$ ($\alpha \neq 0$ will be used below), and hence Proposition A.2 applies and yields

$$\log Z_{n,A'} = \sum_{\mathcal{A} \in \mathfrak{B}_n} w^T(\mathcal{A}) = \sum_{\mathcal{A} \in \mathfrak{B}_n} v^T(\mathcal{A}) 1_{[\mathcal{A} \nsim A']} \tag{4.6}$$

where the truncated weights $v^T(\cdot), w^T(\cdot)$ are related to $v(\cdot), w(\cdot)$, respectively, through the formula (A6) in Section A.2. Comparing to the expansion of $\log Z_{n,\emptyset} = 0$, we get

$$\log Z_{n,A'} = \log \frac{Z_{n,A'}}{Z_{n,\emptyset}} = -\sum_{\mathcal{A} \in \mathfrak{B}_n} v^T(\mathcal{A}) 1_{[\mathcal{A} \sim A']} \tag{4.7}$$

By the bound (A9) in Proposition A.2, applied with $A_0 = A'$, we get immediately

$$|\log Z_{n,A'}| \leq C\epsilon |A'| \tag{4.8}$$

Next, we state

Lemma 4.1. *For $|\lambda|$ small enough and $\kappa = 0$, the following limits exist*

$$z_{\ltimes} := \lim_{n \to \infty} \sum_{A \subset \breve{I}_n : A \ni 0, n+1 \notin A} v(A) Z_{n,A} \tag{4.9}$$

$$z_{\rtimes} := \lim_{n \to \infty} \sum_{A \subset \breve{I}_n : A \ni n+1, 0 \notin A} v(A) Z_{n,A} \tag{4.10}$$

Note that the number $z_{\ltimes} = z_{\ltimes}(\rho_0)$ does not depend on O and $z_{\rtimes} = z_{\rtimes}(O)$ does not depend on ρ_0.

Proof. To deal with the first limit, we recall the (partial) n- independence of polymer weights (2.72), which enables, for $n' > n$,

$$\sum_A^{(n')} v^{(n')}(A) Z_{n',A} - \sum_A^{(n)} v(A) Z_{n,A} = \sum_{A : \max A > n}^{(n')} v^{(n')}(A) Z_{n',A} + \sum_A^{(n)} v(A) (\frac{Z_{n',A}}{Z_{n,A}} - 1) Z_{n,A} \tag{4.11}$$

where we abbreviated $\sum_{A \subset \breve{I}_n : A \ni 0, n+1 \notin A}$ by $\sum_A^{(n)}$ (in particular, the constraint $0 \in A$ is implicit in all terms) and we indicated the n-dependence of polymer weights whenever its omission would cause an ambiguity. We use (4.8) and the bound

$$\sum_{A : d(A) \geq m}^{(n)} |v(A)| \mathrm{e}^{a_v |A|} \leq m^{-\alpha} \breve{C} \breve{\epsilon},$$

which follows from (2.76) in Lemma 2.4, to conclude that the first term in (4.11) is bounded by $\breve{C}\breve{\epsilon}n^{-\alpha}$. To bound the second term, we write, for $\max A \leq n$,

$$|\log \frac{Z_{n',A}}{Z_{n,A}}| \leq \sum_{\mathcal{A}\in\mathfrak{B}_{n'}:\mathcal{A}\sim A} 1_{\max \mathrm{Supp}\mathcal{A}>n}|v^T(\mathcal{A})| \leq C\epsilon(1+n-\max A)^{-\alpha} \tag{4.12}$$

The first inequality follows by inspecting (4.7) and the second follows by the bound 2.75 in Lemma 2.4, which implies decay of cluster weights, as explained in Section A.3. To conclude, we estimate the second term of (4.11) as

$$\sum_{A}^{(n)} |v(A)| \left| \frac{Z_{n',A}}{Z_{n,A}} - 1 \right| |Z_{n,A}| \leq \sum_{A:\max A\leq n/2}^{(n)} \frac{\epsilon C \mathrm{e}^{\epsilon C|A|}|v(A)|}{(1+n-\max A)^{-\alpha}} + \sum_{A:\max A>n/2}^{(n)} |v(A)|\mathrm{e}^{\epsilon C|A|} \tag{4.13}$$

Using again Lemma 2.4, this is bounded by $\breve{C}\breve{\epsilon}n^{-\alpha}$.

The convergence in (4.10) is similar upon recalling the comparison (2.73) between weights of polymers in $\breve{I}_n$ and $\breve{I}_{n'}$. □

In what follows, we now longer trace everywhere the factors $\epsilon, \breve{\epsilon}$, thus making the bounds less sharp then possible.

Lemma 4.2. *For $|\lambda|$ small enough and $\kappa = 0$,*

$$|Z_n(O, \rho_0, \kappa = 0) - k_\ltimes k_\rtimes (1+z_\ltimes)(1+z_\rtimes)| \leq \breve{C}n^{-\alpha} \tag{4.14}$$

and we identify $k_\ltimes(1+z_\ltimes) = \mathrm{Tr}\,\rho_0$.

Proof. We start from (4.2) and we split the sum over $\breve{\mathcal{A}}$ in five parts, of which the first corresponds to $\breve{\mathcal{A}} = \emptyset$. To describe the other parts, let $A_\ltimes$ in general stand for subsets of $\breve{I}_n$ such that $0 \in A, n+1 \notin A$, let $A_\rtimes$ stand for subsets such that $0 \notin A, n+1 \in A$ and, finally, write $A_{\ltimes,\rtimes}$ for subsets such that $\{0, n+1\} \subset A_{\ltimes,\rtimes}$. The splitting is

$$(k_\ltimes k_\rtimes)^{-1} Z_n = Z_{n,\emptyset} + \sum_{A_\ltimes} v(A_\ltimes) Z_{n,A_\ltimes} + \sum_{A_\rtimes} v(A_\rtimes) Z_{n,A_\rtimes} + \sum_{A_\ltimes, A_\rtimes : A_\ltimes \nsim A_\rtimes} v(A_\ltimes) v(A_\rtimes) Z_{n,A_\ltimes \cup A_\rtimes} + \sum_{A_{\ltimes,\rtimes}} v(A_{\ltimes,\rtimes}) Z_{n,A_{\ltimes,\rtimes}} \tag{4.15}$$

We have already argued that the first term on the RHS is 1 and the second and third term converge to $z_\ltimes, z_\rtimes$, respectively. We split the fourth term on the RHS as $\sum_{A_\ltimes, A_\rtimes : A_\ltimes \nsim A_\rtimes} = \sum_{A_\ltimes, A_\rtimes} - \sum_{A_\ltimes, A_\rtimes : A_\ltimes \sim A_\rtimes}$. Let us concentrate on the first term in this splitting, i.e. the unconstrained sum over $A_\ltimes, A_\rtimes$. Clearly, this expression should tend to $z_\ltimes z_\rtimes$. To see this we rewrite using (4.7)

$$\sum_{A_\ltimes, A_\rtimes} v(A_\ltimes) v(A_\rtimes) Z_{n,A_\ltimes \cup A_\rtimes} - \sum_{A_\ltimes, A_\rtimes} v(A_\ltimes) v(A_\rtimes) Z_{n,A_\ltimes} Z_{n,A_\rtimes} = \sum_{A_\ltimes, A_\rtimes} v(A_\ltimes) v(A_\rtimes) Z_{n,A_\ltimes} Z_{n,A_\rtimes} q_n(A_\ltimes, A_\rtimes)$$

with

$$q_n(A_\ltimes, A_\rtimes) = \mathrm{e}^{-\sum_{\mathcal{A}\in\mathfrak{B}_n} v^T(\mathcal{A}) 1_{[\mathcal{A}\sim A_\ltimes]} 1_{[\mathcal{A}\sim A_\rtimes]}} - 1. \tag{4.16}$$

To bound q_n note first that (set $|a|_+ := \max(a, 0)$)

$$\mathrm{dist}(A_\ltimes, A_\rtimes) \geq |n - d(A_\ltimes) - d(A_\rtimes)|_+ \tag{4.17}$$

so that the bound (A13) in Section A.3 together with Lemma 2.4 yields

$$\sum_{\mathcal{A}\in\mathfrak{B}_n} |v^T(\mathcal{A})| 1_{[\mathcal{A}\sim A_\ltimes]} 1_{[\mathcal{A}\sim A_\rtimes]} \leq C(1+|n-d(A_\ltimes)-d(A_\rtimes)|_+)^{-\alpha}\epsilon|A_\ltimes|. \tag{4.18}$$

Using $|\mathrm{e}^{\gamma x} - 1| \leq \gamma \mathrm{e}^{|x|}$ for $0 \leq \gamma \leq 1, x \in \mathbb{R}$ we get from (4.16) and (4.18)

$$|q_n(A_\ltimes, A_\rtimes)| \leq (1+|n-d(A_\ltimes)-d(A_\rtimes)|_+)^{-\alpha} \mathrm{e}^{\epsilon C|A_\ltimes|}. \tag{4.19}$$

Furthermore, since for all $A_\ltimes, A_\rtimes$:

$$(1 + |n - d(A_\ltimes) - d(A_\rtimes)|_+)d(A_\ltimes)d(A_\rtimes) > cn, \tag{4.20}$$

we conclude that

$$|\text{RHS of (4.16)}| \leq Cn^{-\alpha} \left(\sum_{A_\ltimes} d(A_\ltimes)^\alpha |v(A_\ltimes)||Z_{n,A_\ltimes}| \mathrm{e}^{\epsilon C|(A_\ltimes|} \right) \left(\sum_{A_\rtimes} d(A_\rtimes)^\alpha |v(A_\rtimes)||Z_{n,A_\rtimes}| \right) \tag{4.21}$$

The sums between brackets are estimated by $\breve{C}\breve{\epsilon}$ by first using (4.8) and then Lemma 2.4.

Hence, we have obtained the asymptotics of the first part of the fourth term in (4.15):

$$\left| \sum_{A_\ltimes, A_\rtimes} v(A_\ltimes)v(A_\rtimes)Z_{n,A_\ltimes \cup A_\rtimes} - z_\ltimes z_\rtimes \right| \leq \breve{C}n^{-\alpha}. \tag{4.22}$$

By similar, but rather simpler reasoning, we can now treat the other terms in (4.15):

$$\left| \sum_{A_\ltimes} v(A_\ltimes)Z_{n,A_\ltimes} - z_\ltimes \right| \leq \breve{C}n^{-\alpha} \qquad \left| \sum_{A_\rtimes} v(A_\rtimes)Z_{n,A_\rtimes} - z_\rtimes \right| \leq \breve{C}n^{-\alpha} \tag{4.23}$$

$$\left| \sum_{A_\ltimes, A_\rtimes : A_\ltimes \sim A_\rtimes} v(A_\ltimes)v(A_\rtimes)Z_{n,A_\ltimes \cup A_\rtimes} \right| \leq \breve{C}n^{-\alpha} \qquad \left| \sum_{A_{\ltimes,\rtimes}} v(A_{\ltimes,\rtimes})Z_{n,A_{\ltimes,\rtimes}} \right| \leq \breve{C}n^{-\alpha} \tag{4.24}$$

In fact, the upper line has already been obtained in the proof of Lemma 4.1. This proves (4.14).

Take now $O = \mathbb{1}$, then $Z_n = \operatorname{Tr} \rho_0$ and, by the third remark following (2.71) and using that $\tilde{\eta} = \mathbb{1}$ for $\kappa = 0$, we have $k_\rtimes = 1$ and $z_\rtimes = 0$. Then, by the convergence established above, $Z_n \to k_\ltimes(1 + z_\ltimes)$ and hence $\operatorname{Tr} \rho_0 = k_\ltimes(1 + z_\ltimes)$. □

As promised, we comment on the case in which either $k_\ltimes$ or $k_\rtimes$ vanishes, thus invalidating our expression for the polymer weights given in (2.67). Let us assume for concreteness $k_\ltimes = 0$. Then all non-vanishing contributions to (2.71) have $0 \in \mathrm{Supp}\mathcal{A}$. All polymers weights $v(A), A \ni 0$ are obtained by dividing a certain expression by $k_\ltimes$. Therefore, it is possible to redefine $v(A)$ without the division and to omit $k_\ltimes$ in the RHS of (2.71). It is straightforward to check that then the conclusions of the above lemmata still applies when $k_\ltimes(1+z_\ltimes)$ is replaced by $z'_\ltimes$ where $z'_\ltimes$ is defined by removing the division by $k_\ltimes$ in all its terms. The same remark applies to $k_\rtimes$.

4.1.2 Continuity of the polymer representation

We envisage the situation that the model Hamiltonian and/or initial state and observable depend on a parameter $\nu \in \mathcal{D}_\nu \subset \mathbb{C}$ such that our assumptions are satisfied for any $\nu \in \mathcal{D}_\nu$. Note first that the numbers $Z_n(\rho_0, O, \kappa)$ are determined by $a)$ the correlation functions $h, h_\ltimes, h_\rtimes, h_\bowtie$ (we write $\tilde{h}$ to denote any of the four correlation functions), $b)$ the operators $D, H_\mathrm{S}, \rho_{\mathrm{S},0}, O_\mathrm{S}$ in $\mathscr{B}(\mathscr{H}_\mathrm{S})$ (we write X_S to denote any of these operators), and $c)$ the parameter κ. Indeed, the dependence on, for example, the Weyl operators, is hidden in the correlation functions.

Below, we indicate the dependence on the parameter ν by writing $\tilde{h}^{(\nu)}$ and $X_\mathrm{S}^{(\nu)}$.

Lemma 4.3. *Assume that*

1) *The bounds of Assumptions A and C hold with the functions $\tilde{h}(s)$ replaced by $\sup_{\nu \in \mathcal{D}_\nu} |\tilde{h}^{(\nu)}(s)|$.*
2) *Assumption B is satisfied uniformly in $\nu \in \mathcal{D}_\nu$, more precisely, the gap g_T and the constant C_T on the RHS of* (2.49) *can be chosen uniformly in ν.*
3) *The operators $X_\mathrm{S}^{(\nu)}$ are continuous in ν and the functions $\tilde{h}^{(\nu)}$ are pointwise continuous in ν.*

Then, for sufficiently small $|\lambda|, |\kappa|$, the bounds on scalar polymer weights $v(A) = v^{(\nu)}(A), A \in \breve{I}_n$ stated in Lemma 2.4 hold uniformly in ν and $v^{(\nu)}(A)$ is continuous in ν, for any $A \in \breve{I}_n$. Also the factors $k_\ltimes^{(\nu)}, k_\rtimes^{(\nu)}$ are continuous in ν.

Proof. By the dominated convergence theorem, one first establishes that the bounds on $\|G_A^c\|_\diamond$ hold uniformly in ν, and that the operators T_τ, G_A^c are continuous in ν (and the gap of T can be chosen uniform in ν) and then one passes from the operators T_τ, G_A^c to the weights $v^{(\nu)}$. □

Note that we do not claim the continuity to be uniform in λ as $\lambda \to 0$. This can indeed not be deduced because of the rapidly oscillating factors $\mathrm{e}^{\mathrm{i}\lambda^{-2}tH_\mathrm{S}}$ in the definition of G_A, G_A^c.

4.1.3 From discrete to continuous time

We extend the result of Lemma 4.2 to all times t, rather than times of the form $n\lambda^{-2}$. Choose $\ell > 0$ and imagine that we had started our analysis from Hamiltonian $\ell^{-1}H$ instead of H. This would modify the operators X_S and correlation functions $\tilde{h}$ (notation as above) as

$$D^{(\ell)} = \ell^{-1}D, \qquad H_\mathrm{S}^{(\ell)} = \ell^{-1}H_\mathrm{S}, \qquad \tilde{h}^{(\ell)}(s) = \tilde{h}(\ell^{-1}s) \tag{4.25}$$

It is clear that all of our assumptions are satisfied for, say, $\ell \in [1,2]$. Hence all conclusions of our analysis hold for this modified model as well. We focus on $Z_n = Z_n(O, \rho_0, \kappa = 0)$ and let us indicate explicitly the dependence on ℓ by writing $Z_n^{(\ell)}$. By the result of Section 4.1.1, for sufficiently small $|\lambda|$, the limit $Z_\infty^{(\ell)} := \lim_{n\to\infty} Z_n^{(\ell)}$ exists for any $\ell \in [1,2]$. Choose two values ℓ_1, ℓ_2 such that $\ell_1/\ell_2 \in \mathbb{Q}$, then there is a subsequence of $Z_\bullet^{(\ell_1)}$ that coincides with a subsequence of $Z_\bullet^{(\ell_2)}$ and hence $Z_\infty^{(\ell_1)} = Z_\infty^{(\ell_2)}$.

Since the correlation functions $\tilde{h}^{(\ell)}$ are Fourier transforms of L^1-functions, they are pointwise continuous in ℓ. Hence, Lemma 4.3 applies with $\nu \equiv \ell$ and the polymer weights are continuous in $\ell \in [1,2]$. Since the bounds of Section 4.1.1 can be chosen uniform in ℓ, we deduce that $Z_\infty^{(\ell)}$ is continuous in ℓ, as well. Combining this with the arguments above, it follows that $Z_\infty^{(\ell)}$ is in fact independent of ℓ, and we can hence define

$$\langle O\rangle_\infty := (1/\operatorname{Tr}\rho_0) \lim_{t\to\infty} \operatorname{Tr}[O\mathrm{e}^{-\mathrm{i}tH}\rho_0\mathrm{e}^{\mathrm{i}tH}] \tag{4.26}$$

which enables us to proceed to the

4.1.4 Proof of Theorem 1.1

The convergence (4.26) is proven up to now for observables O and initial states ρ_0 satisfying Assumption C. To get point 1) of Theorem 1.1, we show that the convergence holds for any initial density matrix $\rho_0 \in \mathscr{B}_1(\mathscr{H})$ and observable $O \in \mathfrak{W}_\alpha$. First, we note that vectors $\psi_\ltimes$ satisfying Assumption C are dense in $L^2(\mathbb{R}^d)$. By the irreducibility of the Fock representation of the canonical commutation relations, this implies that vectors of the type $\psi_\mathrm{S} \otimes \mathcal{W}(\psi_\ltimes)\Omega$ are dense in $\mathscr{H}$ (see e.g. [25]). Hence, linear combinations of initial density matrices ρ_0 for which (4.26) holds, are dense in $\mathscr{B}_1(\mathscr{H}_\mathrm{S})$. On the other hand, by the unitarity of the dynamics, we have

$$|\operatorname{Tr}(O\rho_t) - \operatorname{Tr}(O\rho_t')| \leq \|O\|_{\mathscr{B}(\mathscr{H})}\|\rho_0 - \rho_0'\|_{\mathscr{B}_1(\mathscr{H})} \tag{4.27}$$

and hence by a simple density argument the convergence (4.26) holds for any $\rho_0 \in \mathscr{B}_1(\mathscr{H})$ and $O \in \mathfrak{W}_\alpha$.

Point 2) of Theorem 1.1. Recall that here O is of the form (1.13) and $\psi_\ltimes$ satisfying (1.15). Without loss, we take $\|O_\mathrm{S}\| = 1$. Then

$$\begin{aligned}
|\langle O\rangle_\infty - \operatorname{Tr}\left[(P_{e_0}\otimes P_\Omega)O\right]| &\leq |\langle O\rangle_\infty - k_\ltimes| + |k_\ltimes - \operatorname{Tr}\left[(P_{e_0}\otimes P_\Omega)O\right]| && (4.28)\\
&\leq |k_\ltimes z_\ltimes| + \|\eta - P_{e_0}\|_{\mathscr{B}_1(\mathscr{H}_\mathrm{S})}\,|\langle\Omega, \mathcal{W}(\psi_\ltimes)\Omega\rangle| && (4.29)\\
&\leq \breve{C}\breve{\epsilon}|k_\ltimes| + C|\lambda|^{2\alpha_*}\,|\langle\Omega, \mathcal{W}(\psi_\ltimes)\Omega\rangle| && (4.30)
\end{aligned}$$

To obtain the second inequality, we used that $\langle O\rangle_\infty = k_\ltimes(1 + z_\ltimes)$, as obtained in Section 4.1.1 and the definition (2.70). The third inequality was established in the proof of Lemma 4.1 (bound on $z_\ltimes$) and in (3.40) (bound on $\eta - P_{e_0}$).

Point 3) of the Theorem was established in Section 4.1.1 for discrete times and is immediately extended to continuous times by the reasoning in Section 4.1.3. Hence the full Theorem 1.1 is now proven.

4.2 Photon number bound

To derive a photon number bound, we are interested in $\mathrm{Tr}[\rho_t \mathrm{e}^{\kappa N}]$, that is, Z_n with $O = \mathbb{1}$ and general ρ_0. However, it is more convenient to investigate first a slightly different quantity; namely

$$p(\kappa) := \lim_{n\to\infty} \frac{1}{n} \log \mathrm{Tr}[(\tilde{\eta} \otimes \mathrm{e}^{\kappa N})\rho_{n/\lambda^2}], \qquad \rho_0 = \eta \otimes P_\Omega \tag{4.31}$$

with $\eta, \tilde{\eta}$ as in (2.50). We call this quantity a 'pressure', since it is the logarithm of Z_n, which we view as a partition function. The advantage of the definition (4.31) is that we can establish right away

Lemma 4.4. *For $|\kappa|$ small enough, the pressure $p(\kappa)$ exists and is analytic in λ*

Proof. Let ρ_{n/λ^2} be as above, then by (2.63) and (2.64),

$$\mathrm{Tr}[(\tilde{\eta} \otimes \mathrm{e}^{\kappa N})\rho_{n/\lambda^2}] = Z_n(\tilde{\eta} \otimes \mathbb{1}, \eta \otimes P_\Omega, \kappa) = \mathrm{Tr}_{\mathrm{S}}\, RQ_{n/\lambda^2}R = \mathrm{e}^{n\theta_T} \sum_{\mathcal{A} \in \mathfrak{B}_n^1} \prod_{A\in\mathcal{A}} v(A) \tag{4.32}$$

Starting from (4.32) and using the cluster expansion results in the Appendix section A.4, we get that $p(\kappa)$ exists for ϵ (hence λ) sufficiently small. To verify that Section A.4 is applicable, we need Lemma 2.4 and the translation invariance of polymer weights, see below (2.71).

We argue that $p(\kappa)$ is analytic in κ by proceeding as in the proof of Lemma 4.3. Note first that the parameter κ enters only (at least for bulk polymers, which are the only ones concerning us here) by multiplying two of the four terms in $K_{u,v}$ by e^{κ}, see (3.5). Hence, by the Vitali convergence theorem, the operators G_A^c, T, and also the scalar polymer weights $v(A)$ are analytic in κ. Since analytic polymer weights imply analyticity of the pressure by the Vitali convergence theorem (see Section A.4.1) the lemma follows. □

Of course, given sufficient infrared regularity, we expect $p(\kappa) = 0$. This will be established in Section 4.2.2 by introducing and removing an infrared cutoff. However, we would like to draw attention to the fact that there is in principle a more straightforward way. One could establish that $\lim_{n\to\infty} \frac{1}{n} \log \mathrm{Tr}[\mathrm{e}^{\kappa N}\rho_{n/\lambda^2}]$ is independent of ρ_0, and then *use the existence of a ground state* Ψ_{gs} with the property that $\langle \Psi_{\mathrm{gs}}, \mathrm{e}^{\kappa N}\Psi_{\mathrm{gs}}\rangle < \infty$ to argue that $\lim_{n\to\infty} \frac{1}{n} \log \mathrm{Tr}[\mathrm{e}^{\kappa N}\rho_{n/\lambda^2}]$ must vanish in general, since it vanishes for $\rho_0 = |\Psi_{\mathrm{gs}}\rangle\langle\Psi_{\mathrm{gs}}|$. We choose not to exploit this approach so as to keep our presentation as self-contained as possible.

Once we have $p(\kappa) = 0$, we still have to exclude that $\mathrm{Tr}[\mathrm{e}^{\kappa N}\rho_t]$ grows subexponentially in t, and this is done in Section 4.2.3.

4.2.1 Infrared cutoff and spectral translations

In this section, we introduce an infrared cutoff γ and we argue that the pressure $p(\kappa, \gamma)$ is continuous in γ.

We implement the cutoff by translating the form factor in the spectral parameter of the one-photon Hamiltonian, i.e. the multiplication with $|q|$ on $L^2(\mathbb{R}, \mathrm{d}q)$. We write $q = \omega\hat{q}$ with $\omega = q$ and $\hat{q} \in \mathbb{S}^{d-1}$, the sphere equipped with the surface measure $\mathrm{d}\hat{q}$. Consider the isomorphism of Hilbert spaces

$$J : L^2(\mathbb{R}^d, \mathrm{d}q) \to L^2(\mathbb{R}_+, L^2(\mathbb{S}^{d-1}, \mathrm{d}\hat{q}), \mathrm{d}\omega), \tag{4.33}$$

such that $(J\psi)(\omega)$ is the function $\hat{q} \to \omega^{\frac{d-1}{2}}\psi(\omega\hat{q})$ in $L^2(\mathbb{S}^{d-1}, \mathrm{d}\hat{q})$. We define the '$\gamma$-translated' form factor $\phi^{(\gamma)}$ by specifying $J\phi^{(\gamma)}$:

$$(J\phi^{(\gamma)})(\omega) := \begin{cases} J\phi(\omega - \gamma) & \omega \geq \gamma \\ 0 & \omega < \gamma \end{cases} \qquad \text{for } \gamma \geq 0 \tag{4.34}$$

In general, the correlation function $h(\cdot)$ can be written as

$$h(t) = \int_{\mathbb{R}^d} \mathrm{d}q \mathrm{e}^{-\mathrm{i}|q|t}|\phi(q)|^2 = \int_{\mathbb{R}^+} \mathrm{d}\omega \mathrm{e}^{-\mathrm{i}\omega t}\|J\phi(\omega)\|_{L^2(\mathbb{S}^{d-1})} \tag{4.35}$$

It is immediate that $h^{(\gamma)}(t) = \mathrm{e}^{-\mathrm{i}\gamma t}h(t)$ and hence in particular

$$\left|h^{(\gamma_1)}(t) - h^{(\gamma_2)}(t)\right| \leq \left|1 - \mathrm{e}^{\mathrm{i}(\gamma_1-\gamma_2)t}\right| |h(t)| \tag{4.36}$$

We want to apply Lemma 4.3 to conclude that the polymer weights $v^{(\gamma)}(A)$ are continuous in γ, and consequently (see Section A.4.1) also $\gamma \mapsto p(\kappa,\gamma)$ is continuous. In fact, the conditions of Lemma 4.3 demand that we also check continuity of the functions $h_{\ltimes}, h_{\gtrdot}, h_{\bowtie}$, but this is not necessary as long as we only use bulk polymers (since those do not depend on $h_{\ltimes}, h_{\gtrdot}, h_{\bowtie}$). Indeed, the pressure $p(\kappa,\gamma)$ above depends only on the bulk polymers and hence we conclude that $p(\kappa,\gamma)$ is continuous in $\gamma \in [0,1]$.

4.2.2 Pressure vanishes

The advantage of having a sharp infrared cutoff is that we can bound the expectation of the number operator in the state Ψ_t in terms of the expectation in the state Ψ_0. Indeed, let us decompose the number operator N as

$$N = N^{\geq\gamma} + N^{<\gamma}, \qquad N^{\geq\gamma} := \int_{|q|\geq\gamma} \mathrm{d}q\, a_q^* a_q, \qquad N^{<\gamma} := \int_{|q|<\gamma} \mathrm{d}q\, a_q^* a_q \tag{4.37}$$

Then because of the cutoff, $[N^{<\gamma}, H] = 0$, and we can bound $N^{\geq\gamma} \leq \gamma^{-1} H_{\mathrm{F}}$. Choosing Ψ_0 such that $\langle\Psi_0, N\Psi_0\rangle < C$ and $\langle\Psi_0, H\Psi_0\rangle < C$ and $\|\Psi_0\| = 1$, we estimate

$$\gamma\langle\Psi_t, N\Psi_t\rangle \quad\leq\quad \gamma\langle\Psi_0, N^{<\gamma}\Psi_0\rangle + \langle\Psi_t, H_{\mathrm{F}}\Psi_t\rangle \tag{4.38}$$

To get an estimate on $\langle\Psi_t, H_{\mathrm{F}}\Psi_t\rangle$, we write $H_{\mathrm{F}} = H - H_{\mathrm{S}} - \lambda H_{\mathrm{I}}$, we use that $\langle\Psi_t, H\Psi_t\rangle = \langle\Psi_0, H\Psi_0\rangle$, the boundedness of H_{S} and the estimate (1.7) to obtain

$$\langle\Psi_t, H_{\mathrm{F}}\Psi_t\rangle \leq C + |\lambda| C' \langle\Psi_t, H_{\mathrm{F}}\Psi_t\rangle \quad\Rightarrow\quad \langle\Psi_t, H_{\mathrm{F}}\Psi_t\rangle \leq C'' \tag{4.39}$$

for λ sufficiently small. By using the spectral theorem and Jensen's inequality, we can now bound $\langle\Psi_t, \mathrm{e}^{\kappa N}\Psi_t\rangle$ with $\kappa \leq 0$ from below and obtain

$$\lim_{n\to\infty} \frac{1}{n} \log\langle\Psi_{\frac{n}{\lambda^2}}, \mathrm{e}^{\kappa N}\Psi_{\frac{n}{\lambda^2}}\rangle = 0, \qquad \kappa \leq 0 \tag{4.40}$$

We want to deduce that also p as defined in (4.31) vanishes. This indeed follows immediately. Firstly, since the operator $\tilde{\eta}$ in the definition of $p(\kappa,\gamma)$ satisfies $\|\mathbb{1} - \tilde{\eta}\| = \mathcal{O}(|\kappa|)$ and for real κ, it is a positive-definite operator, we can find constants c, C such that $c\mathbb{1} \leq \tilde{\eta} \leq C\mathbb{1}$ for sufficiently small, real κ. Secondly, the density matrix $\rho_0 = \eta \otimes P_\Omega$ is a finite sum of terms $|\Psi_0\rangle\langle\Psi_0|$ with Ψ_0 satisfying all conditions necessary for the estimate (4.38). Hence we have shown

$$p(\kappa,\gamma) = 0, \qquad \kappa \leq 0, \gamma > 0. \tag{4.41}$$

However, by analyticity in κ (Lemma 4.4), this holds for sufficiently small $|\kappa|$ and by continuity of the pressure in γ (established in Section 4.2.1), it holds for $\gamma = 0$, as well.

4.2.3 Proof of Theorem 1.3: Finite-size corrections

The fact that $p = 0$ tells us merely that the number of emitted photons grows slower than linearly in time, a result that has been established with other techniques as well. We aim to prove that it does not grow at all. Let us denote the quantitiy in (4.32) by $Z_{n,\emptyset}$, in analogy to the definition in Section 4.1.1 (the only difference is that now $\kappa \neq 0$). We invoke the discussion in Section A.4, in particular the formula (A17) which, applied to the case at hand, reads

$$\log Z_{n,\emptyset} \quad=\quad -\sum_{\mathcal{A}\in\mathfrak{B}_\infty : \min \mathrm{Supp}\mathcal{A} = 1} \min\left(d(\mathcal{A}) - 1, n\right) v^T(\mathcal{A}) \tag{4.42}$$

where we also used that $p = 0$. We bound (4.42) as

$$|\log Z_{n,\emptyset}| \quad\leq\quad Cn^{1-\min(\alpha,1)} \sum_{\mathcal{A}\in\mathfrak{B}_\infty : \min \mathrm{Supp}\mathcal{A} = 1} d(\mathcal{A})^\alpha |v^T(\mathcal{A})| < Cn^{1-\min(\alpha,1)} \tag{4.43}$$

by Proposition A.2. Next, we show that this bound remains valid for arbitrary initial state satisfying Assumption C. We proceed as in Section 4.1.1, obtaining

$$\frac{Z_n}{Z_{n,\emptyset}} = k_{\ltimes} k_{\gtrdot} \sum_{\substack{\breve{\mathcal{A}} \in \breve{\mathfrak{B}}_n^1 \\ \forall A \in \breve{\mathcal{A}} : A \cap \{0, n+1\} \neq \emptyset}} \left(\prod_{A\in\breve{\mathcal{A}}} v(A)\right) \frac{Z_{n,\mathrm{Supp}\breve{\mathcal{A}}}}{Z_{n,\emptyset}} \tag{4.44}$$

In contrast to the situation in Section 4.1.1, $Z_n \neq 1$ since $\kappa \neq 0$. However, the second equality in (4.7) still applies and, in analogy to (4.8) we get $|\log \frac{Z_{n,\mathrm{Supp}\breve{\mathcal{A}}}}{Z_{n,\emptyset}}| \leq \epsilon C |\mathrm{Supp}\breve{\mathcal{A}}|$ and then also

$$|\frac{Z_n}{Z_{n,\emptyset}}| \leq \breve{C}, \tag{4.45}$$

by combining (4.44) and (2.76) and making ϵ sufficiently small.

Hence we have in general

$$|Z_n| = |\frac{Z_n}{Z_{n,\emptyset}}||Z_{n,\emptyset}| \leq \breve{C}\mathrm{e}^{Cn^{1-\min(\alpha,1)}} \tag{4.46}$$

Finally, we mimic the reasoning in Section 4.1.3 to extend the bound (4.46) to continuous time. This finishes the proof of Theorem 1.3.

A Cluster Expansions

We present some standard results on cluster expansions. We opted not to present a very general exposition, but instead a framework that is maximally adapted to our problem. A slight exception is Section A.1 where we state a combinatorial lemma that is used in several places in our paper. In that section, the 'polymers' can be anything, but in the remaining sections of this appendix, we assume them to be subsets A of $\mathbb{N}$, just as in our construction, and although we call the polymer weights $w(A)$, the reader can safely confuse them with our concrete polymer weights $v(A)$ defined in Section 2.3. In the sections other than Section A.1, we also freely use the notation introduced in Section 2.1.5. This appendix follows very closely the presentation in [30], which uses the convergence criterion by [22].

A.1 A combinatorial lemma

Let $\mathscr{S}$ be a countable set. We call its elements $S \in \mathscr{S}$ *polymers*. A function $w : \mathscr{S} \to \mathbb{C}$ will be called a *polymer weight*. Furthermore, let $\sim$ be an adjacency relation on $\mathscr{S}$ i.e. $\sim$ is a symmetric and irreflexive relation. Given a finite collection of polymers $\mathcal{S} \subset \mathscr{S}$ we define

$$k(\mathcal{S}) := \sum_{\mathscr{G} \in \mathfrak{G}^c(\mathcal{S})} \prod_{\{S_i,S_j\} \in \mathscr{E}(\mathscr{G})} 1_{[S_i \sim S_j]} \tag{A1}$$

where the sum runs over $\mathfrak{G}^c(\mathcal{S})$, the set of connected graphs with vertex set $\mathcal{S}$, and $\mathscr{E}(\mathscr{G})$ is the edge set of the graph $\mathscr{G}$. Finally, for a collection $\mathcal{S}$ and a polymer S_0, we write $\mathcal{S} \sim S_0$ whenever there is a $S \in \mathcal{S}$ such that $S \sim S_0$. Then, we have the following combinatorial result

Lemma A.1. *Assume that there is a positive function $S \mapsto a(S)$ such that the so-called "Kotecky-Preiss criterion"*

$$\sum_{S : S \sim S'} |w(S)| \mathrm{e}^{a(S)} \leq a(S') \tag{A2}$$

holds. Then

$$\sum_{\mathcal{S} : \mathcal{S} \sim S_0} k(\mathcal{S}) \prod_{S \in \mathcal{S}} |w(S)| \leq a(S_0) \tag{A3}$$

and

$$\sum_{\mathcal{S}} k(\mathcal{S} \cup \{S_0\}) \prod_{S \in \mathcal{S}} |w(S)| \leq \mathrm{e}^{a(S_0)} \tag{A4}$$

where the term corresponding to $\mathcal{S} = \emptyset$ is understood to equal 1.

An easy way to prove (A4) is via induction in the number of elements of the collections $\mathcal{S}$. The claim (A3) then follows by choosing $S_1 \in \mathcal{S}$ such that $S_0 \sim S_1$ and applying (A4) with S_1 in the role of S_0.

A.2 Logarithm of the partition function

We make the setup introduced in Section A.1 more concrete

1) The polymers are now subsets of $I_n = \{1, \dots, n\}$. They are denoted by $A, A', \dots$.
2) The adjacency relation $\sim$ is defined to be $A \sim A' \Leftrightarrow \mathrm{dist}(A, A') \leq 1$ with $\mathrm{dist}(A, A') = \inf_{\tau \in A, \tau' \in A'} |\tau - \tau'|$.
3) Collections of polymers are denoted by $\mathcal{A}$. We use the sets of collections $\mathfrak{B}_n, \mathfrak{B}_n^1$ introduced in Section 2.1.5.

We start from a polymer-gas representation of some *partition function* Υ_n:

$$\Upsilon_n = \sum_{\mathcal{A} \in \mathfrak{B}_n^1} \prod_{A \in \mathcal{A}} w(A) = \sum_{\mathcal{A} \in \mathfrak{B}_n} \prod_{A \in \mathcal{A}} w(A) \prod_{\{A_i, A_j\} \subset \mathcal{A}} 1_{A_i \nsim A_j} \tag{A5}$$

where the last product runs over pairs in the collection $\mathcal{A}$, and the summand is defined to be 1 if $\mathcal{A} = \emptyset$.

We define the *truncated weights* w^T of $\mathcal{A} \in \mathfrak{B}_n$ as follows

$$w^T(\mathcal{A}) := \sum_{\mathscr{G} \in \mathfrak{G}^c(\mathcal{A})} (-1)^{|\mathscr{E}(\mathscr{G})|} \prod_{\{A_i, A_j\} \in \mathscr{E}(\mathscr{G})} 1_{[A_i \sim A_j]} \prod_{A_i \in \mathcal{A}} w(A) \tag{A6}$$

where the sum runs over $\mathfrak{G}^c(\mathcal{A})$, the set of connected graphs with vertex set $\mathcal{A}$, and $\mathscr{E}(\mathscr{G})$ is the edge set of the graph $\mathscr{G}$. We call $\mathcal{A}$ a cluster whenever the graph on $\mathcal{A}$ with edge set $\{\{A_i, A_j\}, A_i \sim A_j\}$ is connected. Hence, if $\mathcal{A}$ is not a cluster, then $w^T(\mathcal{A}) = 0$.

Next, we state the basic result of cluster expansions (cfr. (eq. 4) in [30]).

Proposition A.2. *Assume there is $\delta > 0$ such that the 'Kotecky-Preiss criterion'*

$$\sum_{A \subset I_n : A \sim A'} \mathrm{e}^{\delta |A|} w(A) \leq \delta |A'| \tag{A7}$$

holds. Then $\Upsilon_n \neq 0$ and

$$\log \Upsilon_n = \sum_{\mathcal{A} \in \mathfrak{B}_n} w^T(\mathcal{A}), \tag{A8}$$

and, for any $A_0 \subset I_n$

$$\sum_{\mathcal{A} \in \mathfrak{B}_n : \mathcal{A} \sim A_0} \left| w^T(\mathcal{A}) \right| \leq \delta |A_0| \tag{A9}$$

This proposition is a direct consequence of Lemma A.1. It is clear how it applies to the present paper. The polymer weights $w(A)$ are $v(A)$ as constructed in Section 2.3, and the Kotecky-Preiss criterion is (2.75) where δ should be chosen not smaller than $C\epsilon$ and not larger than c (C, c as in (2.75)).

A.3 Decay of cluster weights

We assume now that the Kotecky-Preiss criterion is satisfied in a stronger sense, namely:

$$\sum_{A \subset I_n : A \sim A'} \mathrm{e}^{\delta |A|} w_\alpha(A) \leq \delta |A'|, \tag{A10}$$

where

$$w_\alpha(A) := d(A)^\alpha w(A)$$

and we show that this yields some decay in the cluster weights. Proposition A.2 gives

$$\sum_{\mathcal{A} \in \mathfrak{B}_n : \mathcal{A} \sim A_0} \left| w_\alpha^T(\mathcal{A}) \right| = \sum_{\mathcal{A} \in \mathfrak{B}_n : \mathcal{A} \sim A_0} \prod_{A \in \mathcal{A}} d(A)^\alpha w^T(\mathcal{A}) \leq \delta |A_0| \tag{A11}$$

If $\mathcal{A}$ is a cluster then $\sum_{A \in \mathcal{A}} d(A) \geq d(\mathcal{A})$, and if additionally all polymers A with nonzero weight have $d(A) > 1$ (this is our case), then

$$\prod_{A \in \mathcal{A}} d(A)^\alpha \geq d(\mathcal{A})^\alpha \tag{A12}$$

Hence, we use (A11) to derive

$$\sum_{\mathcal{A}\in\mathfrak{B}_n:\mathcal{A}\sim A_0} 1_{d(\mathcal{A})\geq m}\left|w^T(\mathcal{A})\right| \quad\leq\quad C\delta|A_0|(1+m)^{-\alpha} \tag{A13}$$

A.4 Pressure

In the previous sections, n was a fixed parameter. Now, we consider different values of n. If $A\subset I_n\subset I_{n'}$ ($n'>n$), we assume that $w(A)$ does not depend on whether we view A as polymer in I_n or $I_{n'}$. Moreover, we assume that the weights are translation-invariant in the sense that

$$w(A)=w(A+a) \tag{A14}$$

where $A+a:=\{\tau,\tau-a\in A\}$. An immediate consequence is that the pressure

$$p:=\lim_{n\to\infty} n^{-1}\log\Upsilon_n \tag{A15}$$

exists and is given by

$$p=\sum_{\mathcal{A}\in\mathfrak{B}_\infty,\min\mathrm{Supp}\mathcal{A}=1} w^T(\mathcal{A}) \tag{A16}$$

where $\mathfrak{B}_\infty=\cup_n\mathfrak{B}_n$ is the set of finite collections of finite subsets of $\mathbb{N}$. Indeed, the absolute summability of the RHS of (A16) follows from Proposition A.2, whereas (we set $|a|_+:=\max(a,0)$)

$$p-n^{-1}\log\Upsilon_n \quad=\quad \sum_{\mathcal{A}\in\mathfrak{B}_\infty:\min\mathrm{Supp}\mathcal{A}=1}\left(\frac{n-|n+1-d(\mathcal{A})|_+}{n}\right)w^T(\mathcal{A}) \tag{A17}$$

which vanishes as $n\to\infty$ be the dominated convergence theorem, since the factor between brackets converges to 0 for any value of $d(\mathcal{A})$.

A.4.1 Continuity and analyticity of the pressure

Often, one encounters the situation where the weights $w(A)$ depends on a parameter $\nu\in\mathcal{D}_\nu\subset\mathbb{C}$, $w(A)=w^{(\nu)}(A)$ such that the Kotecky-Preiss criterion (A7) is satisfied even when $w(A)$ is replaced by $\sup_{\nu\in\mathcal{D}_\nu}|w^{(\nu)}(A)|$. Then

1) If the weights $w^{(\nu)}(A)$ are continuous in $\nu\in\mathcal{D}_\nu$, for any $A\subset\mathbb{N}$, then the pressure is continuous in $\nu\in\mathcal{D}_\nu$.
2) If the weights $w^{(\nu)}(A)$ are analytic in $\nu\in\mathcal{D}_\nu$, for any $A\subset\mathbb{N}$, then the pressure is analytic in $\nu\in\mathcal{D}_\nu$.

Starting from the absolute summability of the RHS of (A16), the first claim follows from the dominated convergence theorem, and the second from the Vitali convergence theorem.

B Van Hove limit

In Section 3.5, we introduced the Lindblad generator M and we stated and exploited the claim (Proposition 3.3) that

$$\|Q_t-\mathrm{e}^{-\mathrm{i}tL_{\mathrm{S}}+\lambda^2tM}\|\leq\mathrm{e}^{C\lambda^2t}|\lambda|^{2\alpha_*} \tag{B1}$$

Below we derive this claim. The derivation is independent of the rest of the paper, except for the representation of the effective dynamics Q_t:

$$Q_t=\int_{\Sigma_{[0,t]}}\mu(\mathrm{d}\underline{w})\,\mathcal{T}\left\{\bigotimes_{w\in\underline{w}}K_w\right\} \tag{B2}$$

and the bound $\|K_{u,v}\|_\diamond\leq\lambda^2|h(v-u)|$. Moreover, we freely use Assumption A, i.e. $\int_0^\infty\mathrm{d}t(1+t)^\alpha|h(t)|\leq C$. We set $\kappa=0$ in this Appendix. Including a nonzero value for κ does not affect the reasoning.

B.1 Alternative construction of M

We define the operator $F_s \in \mathscr{R} = \mathscr{B}(\mathscr{B}_1(\mathscr{H}_{\rm S}))$ using the notation of Section 3;

$$\lambda^2 F_{v-u} \;:=\; \mathrm{e}^{-\mathrm{i}vL_{\rm S}}\mathcal{T}[K_{u,v}]\mathrm{e}^{\mathrm{i}uL_{\rm S}} \tag{B3}$$

For the sake of clarity, we write out F_s explicitly (recall that $\kappa = 0$ here)

$$\begin{aligned} F_s &= -h(s)\mathcal{R}(D)\mathrm{e}^{-\mathrm{i}sL_{\rm S}}\mathcal{L}(D) - h(-s)\mathcal{L}(D)\mathrm{e}^{-\mathrm{i}sL_{\rm S}}\mathcal{R}(D) \\ &+ h(s)\mathcal{L}(D)\mathrm{e}^{-\mathrm{i}sL_{\rm S}}\mathcal{L}(D) + h(-s)\mathcal{R}(D)\mathrm{e}^{-\mathrm{i}sL_{\rm S}}\mathcal{R}(D) \end{aligned} \tag{B4}$$

Introduce spectral projections $P_{\varepsilon\in\mathcal{E}}$ of the Liouvillian $L_{\rm S}$. The range of $P_{\varepsilon=0}$ has dimension $\dim \mathscr{H}_{\rm S}$ and the $P_{\varepsilon\neq 0}$ are one-dimensional. The connection between the operators F_s and the generator M given in Section 3.5 is

$$M \;=\; \sum_{\varepsilon\in\mathcal{E}}\int_0^\infty \mathrm{d}s\,\mathrm{e}^{\mathrm{i}s\varepsilon}P_\varepsilon F_s P_\varepsilon \tag{B5}$$

This can be checked by direct computation. The four terms in (B4) are matched with the expression (3.34) as follows: the first two terms give the first term under the integral (with κ). The two other terms yield the commutator with H_{lamb} and the second term under the integral. Note also that the part of M corresponding to the $\varepsilon = 0$ term on the RHS of (B5) is isomorphic to the operator $\mathcal{M}$ discussed in Section 3.5.1, and the parts corresponding to $\varepsilon \neq 0$ were discussed under the heading 'off-diagonal elements' in that same section.

B.2 Emergence from the Dyson expansion

We start from the Dyson series (B2) and we split the set of collections of pairs $\Sigma_{[0,t]}$ into leading $\Sigma^{\rm l}_{[0,t]}$ and non-leading $\Sigma^{\rm nl}_{[0,t]}$ subsets, as follows

$$\Sigma^{\rm l}_{[0,t]} := \{\underline{w} \in \Sigma_{[0,t]},\, \forall i = 1,\dots,|\underline{w}|-1 : v_i < u_{i+1}\}, \qquad \Sigma^{\rm nl}_{[0,t]} := \Sigma_{[0,t]} \setminus \Sigma^{\rm l}_{[0,t]} \tag{B6}$$

(note that $\emptyset \in \Sigma^{\rm l}_{[0,t]}$). Define

$$\begin{aligned} Q_{t,\rm l} \;&:=\; \int\limits_{\Sigma^{\rm l}_{[0,t]}} \mu(\mathrm{d}\underline{w})\,\mathcal{T}\left\{\otimes_{w\in\underline{w}}K_w\right\} \\ &=\; \sum_{m=0}^{\infty}\lambda^{2m}\int\limits_{0<\ldots<u_i<v_i<u_{i+1}<\ldots<t} \mathrm{d}\underline{u}\mathrm{d}\underline{v}\,\mathrm{e}^{-\mathrm{i}(t-v_m)L_{\rm S}}F_{v_m-u_m}\ldots F_{v_2-u_2}\mathrm{e}^{-\mathrm{i}(u_2-v_1)L_{\rm S}}F_{v_1-u_1}\mathrm{e}^{-\mathrm{i}u_1L_{\rm S}} \end{aligned}$$

Its Laplace transform is

$$\hat{Q}_{z,\rm l} := \int_{\mathbb{R}_+}\mathrm{d}t\,\mathrm{e}^{-tz}Q_{t,\rm l} = \left(z + \mathrm{i}L_{\rm S} - \lambda^2\hat{F}_z\right)^{-1} \tag{B7}$$

where $\hat{F}_z = \int_{\mathbb{R}_+}\mathrm{d}t\,\mathrm{e}^{-tz}F_t$. By inverting the Laplace transform we get

Lemma B.1.

$$\left\|Q_{t,\rm l} - \mathrm{e}^{-\mathrm{i}tL_{\rm S}+\lambda^2 tM}\right\| \leq \mathrm{e}^{\lambda^2 tC}C|\lambda|^{2\alpha_*} \tag{B8}$$

where $\alpha_* = \min(1,\alpha)$ *for* $\alpha \neq 1$, *and* $|\lambda|^{2\alpha_*} = \lambda^2|\log|\lambda||$ *for* $\alpha = 1$.

Proof. We start by writing

$$Q_{t,\rm l} - \mathrm{e}^{-\mathrm{i}tL_{\rm S}+\lambda^2 tM} = \int_{\mathcal{C}}\mathrm{d}z(\hat{Q}_{z,\rm l} - \hat{Q}_{z,0})\mathrm{e}^{tz} \tag{B9}$$

where $\mathcal{C}$ is a contour in the complex plane, of the form $\mathcal{C} = \mathrm{i}\mathbb{R} + A\lambda^2$ with A large enough and

$$\hat{Q}_{z,0} = (z + \mathrm{i}L_{\rm S} - \lambda^2 M)^{-1}.$$

Indeed, since the spectrum $\mathcal{E}$ of $L_{\rm S}$ is real and $\|\hat{F}_z\|$ is uniformly bounded for $\mathrm{Re}\, z \geq 0$, the function $\hat{Q}_{z,l}$ is analytic in the region $\mathrm{Re}\, z \geq A\lambda^2$ if A is large enough and so (B9) holds.

Next, simple algebra yields

$$\hat{Q}_{z,1} - \hat{Q}_{z,0} = \lambda^2 \hat{Q}_{z,1}(\hat{F}_z - M)\hat{Q}_{z,0} \tag{B10}$$

and from (B5)

$$M = \sum_{\varepsilon} P_\varepsilon \hat{F}_{-\mathrm{i}\varepsilon} P_\varepsilon. \tag{B11}$$

We decompose $\int_{\mathcal{C}} = \sum_{\varepsilon\in\mathcal{E}} \int_{\mathcal{C}_\varepsilon}$ where $\mathcal{C}_\varepsilon := \{z \in \mathcal{C}, |z + \mathrm{i}\varepsilon| = \min_{\varepsilon'\in\mathcal{E}} |z + \mathrm{i}\varepsilon'|\}$. For $z \in \mathcal{C}_\varepsilon$ we write

$$\hat{F}_z - M = \left(\hat{F}_{-\mathrm{i}\varepsilon} - M\right) + \left(\hat{F}_z - \hat{F}_{-\mathrm{i}\varepsilon}\right) := V_\varepsilon + W_\varepsilon. \tag{B12}$$

The lemma follows if we show

$$\lambda^2 \sum_{\varepsilon\in\mathcal{E}} \int_{\mathcal{C}_\varepsilon} \mathrm{d}z \|\hat{Q}_{z,1}(V_\varepsilon + W_\varepsilon)\hat{Q}_{z,0}\| \leq C\lambda^{2\alpha_*}. \tag{B13}$$

We need the bounds, for $z \in \mathcal{C}$,

$$\|\hat{Q}_{z,j}\| \leq C(\mathrm{dist}(z, -\mathrm{i}\mathcal{E}))^{-1} \tag{B14}$$

$$\|\hat{Q}_{z,j}P_\varepsilon\| \leq C|z + \mathrm{i}\varepsilon|^{-1} \tag{B15}$$

for $j = 1, 0$. The bound for $j = 1$ follow by straightforward estimates on the Neumann series

$$\left(z + \mathrm{i}L_{\rm S} - \lambda^2\hat{F}_z\right)^{-1} = (z + \mathrm{i}L_{\rm S})^{-1} \sum_{n=0}^{\infty} \left(\lambda^2 \hat{F}_z\, (z + \mathrm{i}L_{\rm S})^{-1}\right)^n \tag{B16}$$

using in particular $\|(z + \mathrm{i}L_{\rm S})^{-1}\| \leq C(\mathrm{dist}(z, -\mathrm{i}\mathcal{E}))^{-1} \leq C|\lambda|^{-2}$. The $j = 0$ case follows similarly. Using (B11) we have

$$V_\varepsilon = \sum_{(\varepsilon',\varepsilon'')\neq(\varepsilon,\varepsilon)} P_{\varepsilon'} \hat{F}_{-\mathrm{i}\varepsilon} P_{\varepsilon''}$$

so combining with (B14) and (B15)

$$\|\hat{Q}_{z,1}V_\varepsilon\hat{Q}_{z,0}\| \leq C \sum_{\varepsilon'\neq\varepsilon} \frac{1}{|z + \mathrm{i}\varepsilon||z + \mathrm{i}\varepsilon'|}, \qquad \text{for } z \in \mathcal{C}_\varepsilon.$$

The V_ε contribution to (B13) is then bounded by

$$C\lambda^2 \sum_{\varepsilon'\neq\varepsilon} \int_{\mathcal{C}_\varepsilon} \mathrm{d}z\, \frac{1}{|z + \mathrm{i}\varepsilon||z + \mathrm{i}\varepsilon'|} < C\lambda^2 |\log|\lambda||. \tag{B17}$$

As for the W_ε we have

$$\|\hat{F}_z - \hat{F}_{-i\varepsilon}\| \leq \int_0^\infty \mathrm{d}t\, h(t) \left|\mathrm{e}^{-zt} - \mathrm{e}^{i\varepsilon t}\right|. \tag{B18}$$

Using $\int_0^\infty \mathrm{d}t\, h(t)(1 + t)^\alpha \leq C$, we obtain for $\mathrm{Re}\, z \geq 0$

$$\|\hat{F}_z - \hat{F}_{-i\varepsilon}\| \leq C \min\{|z + i\varepsilon|^{\min(\alpha,1)}, 1\}. \tag{B19}$$

Hence the W_ε contribution to (B13) is bounded by

$$C\lambda^2 \sum_{\varepsilon} \int_{\mathcal{C}_\varepsilon} \mathrm{d}z\, \frac{1}{|z + \mathrm{i}\varepsilon|^2}\, \min(1, |z + i\varepsilon|^{\min(\alpha,1)}) \leq C|\lambda|^{2\alpha_*}. \tag{B20}$$

(B13) follows now from (B17) and (B20). □

It remains to estimate the contribution of the non-leading pairs:

Lemma B.2. *For any* $\mathfrak{t} > 0$*;*

$$\sup_{0<t<\lambda^{-2}\mathfrak{t}} \int\limits_{\Sigma^{\mathrm{nl}}_{[0,t]}} \mathrm{d}\underline{w} \, \| \mathcal{T}\left\{\otimes_{w\in\underline{w}} K_w\right\} \| \leq \mathrm{e}^{\mathfrak{t}C}|\lambda|^{2\min(1,\alpha)} \tag{B21}$$

Proof. We start with a bound analogous to (3.20):

$$\int\limits_{\Sigma^{\mathrm{nl}}_{[0,t]}} \mathrm{d}\underline{w} \, \| \mathcal{T}\left\{\otimes_{w\in\underline{w}} K_w\right\} \| \leq \int\limits_{\Sigma^{\mathrm{nl}}_{[0,t]}} \mathrm{d}\underline{w} \prod_{w\in\underline{w}} \lambda^2 C|h(v-u)| \tag{B22}$$

By definition, every $\underline{w} \in \Sigma^{\mathrm{nl}}_{[0,t]}$ has to contain at least two pairs $(u,v),(u',v')$ such that $u < u' < v$. Choose the first two such pairs and integrate over the coordinates of all other pairs, proceeding as in (3.24). This yields

$$\int\limits_{\Sigma^{\mathrm{nl}}_{[0,t]}} \mathrm{d}\underline{w} \prod_{w\in\underline{w}} C\lambda^2|h(v-u)| \leq C\mathrm{e}^{\lambda^2 Ct\|h\|_1}\lambda^4 q(t) \tag{B23}$$

with

$$q(t) = \int\limits_{0<u<u'<v<t,u'<v'<t} \mathrm{d}u\mathrm{d}v\mathrm{d}u'\mathrm{d}v' \, |h(v-u)h(v'-u')|. \tag{B24}$$

$q(t)$ is estimated by first performing the integral over v', which gives a factor $\|h\|_1$, and then the one over u', giving $|v-u|$:

$$\begin{aligned} q(t) &\leq \|h\|_1 \int\limits_{0<u<v<t} \mathrm{d}u\mathrm{d}v|h(v-u)||v-u| \\ &\leq \|h\|_1 \int_0^t \mathrm{d}u \left(\max_{0<s<t-u}(1+|s|)^{1-\alpha}\right) \int_0^\infty \mathrm{d}s \, (1+|s|)^\alpha |h(s)| \\ &\leq \|h\|_1 C t^{2-2\min(1,\alpha)} \end{aligned} \tag{B25}$$

Taking $0 < t \leq |\lambda|^{-2}\mathfrak{t}$, this yields the lemma. □

The bound (B1) (and hence Proposition 3.3) follows by combining Lemmata B.2 and B.1.

C Infrared Regularity

We clarify how our Assumption A, formulated in the time-domain, relates to infrared regularity of the form factor ϕ. First, we state the auxiliary

Lemma C.1. *Let* $f : (0,\infty) \to \mathbb{R}$ *be a measurable function of compact support and such that for some* $0 < \gamma < 1$*,*

$$|\partial^n f(\omega)| \leq C_0 \omega^{\gamma-1-n}, \qquad n = 0, 1 \tag{C1}$$

Then, for any $\beta < \gamma$*,*

$$\left| \int_0^\infty f(\omega)\mathrm{e}^{\mathrm{i}t\omega}\mathrm{d}\omega \right| \leq C(\beta)|t|^{-\beta}$$

Proof. Define, for $0 < \beta \leq 1$,

$$|f|_\beta := \int_0^\infty \mathrm{d}\omega \sup_{0<\epsilon\leq 1} \epsilon^{-\beta}|f(\omega+\epsilon) - f(\omega)| < \infty \tag{C2}$$

Then, we claim that

$$\left|\int_0^\infty f(\omega)e^{it\omega}\mathrm{d}\omega\right| \leq \frac{1}{|1-\mathrm{e}^{-\mathrm{i}}|}\left(|f|_\beta|t|^{-\beta}+C\gamma^{-1}|t|^{-\gamma}\right) \tag{C3}$$

Indeed, writing (say $t>0$, the case $t<0$ follows by replacing f by $\bar{f}$)

$$\int_0^\infty f(\omega)\mathrm{e}^{\mathrm{i}t\omega}\mathrm{d}\omega = \frac{t^{-\beta}}{\mathrm{e}^{-\mathrm{i}}-1}\int_0^\infty \frac{f(\omega+1/t)-f(\omega)}{t^{-\beta}}\mathrm{e}^{\mathrm{i}t\omega}d\omega + \frac{\mathrm{e}^{-\mathrm{i}}}{\mathrm{e}^{-\mathrm{i}}-1}\int_0^{1/t} f(\omega)\mathrm{e}^{\mathrm{i}t\omega}\mathrm{d}\omega$$

the claim follows. To verify that $|f|_\beta<\infty$ for any $\beta<\gamma$, we estimate

$$\epsilon^{-\beta}|f(\omega+\epsilon)-f(\omega)| \leq \begin{cases} \omega^{-\beta}\left(|f(\omega)|+|f(\omega+\epsilon)|\right) & \omega\leq\epsilon \\ \omega^{1-\beta}\sup_{\omega'\in[\omega,\omega+\epsilon]}|\partial f(\omega')| & \omega>\epsilon \end{cases}$$

and then use (C1) to bound this by $C\omega^{-\beta+\gamma-1}$, which is integrable for any $\beta<\gamma$. □

Using the above Lemma, we can state a sufficient condition on the form factor ϕ for Assumption A to hold.

Lemma C.2. *Fix $\gamma>0$ (not necessarily $\gamma<1$) and let $\lfloor\gamma\rfloor$ be the highest integer not larger than γ. Let $f_{\hat{q}}(\omega):=\omega^{d-1}|\phi(\omega\hat{q})|^2$, with $\hat{q}\in\mathbb{S}^{d-1}$ and assume that*

$$\int_1^\infty |\partial_\omega^n f_{\hat{q}}(\omega)|\leq C, \qquad \text{and} \qquad \forall\omega\in(0,2]:\ |\partial_\omega^n f_{\hat{q}}(\omega)|\leq C\omega^{\gamma-n}, \tag{C4}$$

for any $n=0,1,\ldots,\lfloor\gamma\rfloor+2$ and uniformly in $\hat{q}$. Then, Assumption A holds with $\alpha<\gamma$.

Proof. Let $\theta:\mathbb{R}^+\to[0,1]$ be a C^∞ function such that

$$\theta(\omega)=\begin{cases}1 & \omega\leq 1\\ 0 & \omega\geq 2\end{cases} \tag{C5}$$

and write $f=f^{(1)}+f^{(2)}$ with $f^{(1)}=\theta f$ and $f^{(2)}=(1-\theta)f$ where we omit the parameter $\hat{q}$ (all bounds are uniform in $\hat{q}$). Then $\int\mathrm{d}\omega f^{(2)}(\omega)\mathrm{e}^{-\mathrm{i}t\omega}$ decays as $t^{-\lfloor\gamma\rfloor+2}$ by the first bound in (C4) and partial integration. To estimate $\int\mathrm{d}\omega f^{(1)}(\omega)\mathrm{e}^{-\mathrm{i}t\omega}$, we use the second bound in (C4) and partial integration $\lfloor\gamma\rfloor+1$ times to extract a factor $t^{-(\lfloor\gamma\rfloor+1)}$. Then we use Lemma C.1 but with $\gamma-\lfloor\gamma\rfloor\leq 1$ playing the role of γ in Lemma C.1, to extract the additional decay $t^{-\beta}$ with $\beta<\gamma-\lfloor\gamma\rfloor$. □

Likewise, one can derive from Lemma C.1 sufficient conditions on $\phi,\psi_\ltimes,\psi_\rtimes$ to satisfy Assumption C.

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